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FINAL REPORT*

RESEARCH PROJECT DAERO-77-G-056

GEODETIC APPLICATIONS OF INERTIAL NAVIGATIONAL SYSTEMS

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A gradiometer-aided inertial navigation system is theoretically and statistically analysed to estimate its abilities to monitore geocentric cartesian coordinates. Having discussed the inertial instrumental units used on the moving platform and several reference coordinate frames applicable in all navigation systems, studies on the severe problem of the separability of the gravity gradients from the inertial disturbances are carried out. Simulation I presents how well the aided navigation system can produce inertial co- → next page

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ordinates and how the newcomers of the inertial instrumentation, the gradiometers, perform on-board the moving vehicle. Quantization error studies are also analysed and presented for such a system. Simulation II includes besides the detailed analysis of the accelerometer and gradiometer error models used, the abilities of the system to estimate geocentric coordinates. Multipoint statistical analysis for the approximated inertial acceleration components shows that the navigation system under consideration behaves better as closer the reality is approached.

Final Report
Research Project DAERO-77-G-056
Geodetic Applications of Inertial Navigational Systems

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Reference is made to the mid-term report. In addition, the simulation study I has been completed by a more general model in simulation study II. A detailed report of 115 pages is added to this summary.

The new analysis of geodetic applications of inertial navigation systems started from the basic equation

"observed acceleration =
inertial acceleration minus gravitation"

which can be solved for inertial acceleration if we know gravitation, e.g. from integrated gravity gradient measurements. The naive approach is as following: Measure the coordinates of the apparent acceleration vector, e.g. A_x, A_y, A_z, and the coordinates of the tensor of gravity gradients, e.g. G_{xx}, G_{xy}, G_{xz}, G_{yy}, G_{yz}, G_{zz}. Then compute at point one

$$A_{x_1} + G_{x_o} + G_{xx_o}(X_1 - X_o) + G_{xy_o}(Y_1 - Y_o) + G_{xz_o}(Z_1 - Z_o) =$$

$$\Delta t^{-2}(X_2 - 2X_1 + X_o)$$

A_{y_1}, A_{z_1} analogous

where Δt indicates the time interval of measurements; the right side approximates inertial acceleration by Stirling's formula.

What we have done at point one can be done at any point such that

$$X_{i+1} = \Delta t^2\Big[A_{x_i} + G_{x_i} + G_{xx_{i-1}}(X_i - X_{i-1}) + G_{xy_{i-1}}(Y_i - Y_{i-1})$$
$$+ G_{xz_{i-1}}(Z_i - Z_{i-1})\Big] + \text{quantization errors}$$

Y_{i+1}, Z_{i+1} analogous

are the unknown coordinates of points with respect to inertial space we like to know. The set-up is in terms of an initial value problem since we have to know (X_o, Y_o, Z_o) (X_1, Y_1, Z_1), the coordinates of the initial position vectors, and $(G_{x_o}, G_{y_o}, G_{z_o})$, the coordinates of the initial gravity vector. In addition, the Laplacean

$$G_{xx} + G_{yy} + G_{zz} = 0$$

is a physical condition if we measure outside the masses (surfaces included). Finally, a transformation of inertial coordinates into terrestrial ones has to be performed.

Simulation I is an error budget study by 19 parameters of an inertial system:

(i) time interval
(ii) initial positions
(iii) initial gravity
(iv) varying acceleration
(v) varying gravity gradients

The input-output results are given on pages 46 - 56 of the detailed report. Pages 61 - 67 are a study of the influence of quantization errors. This is a summary:
The gradiometer accuracy of existing systems is by far sufficient for inertial navigation applications; the main error budget is due to the accelerometer accuracy.

Simulation II is an error budget study by 36 parameters of an inertial system:

(i) time interval
(ii) initial positions
(iii) initial gravity
(iv) varying acceleration
(v) varying gravity gradients
(vi) accelerometer bias
(vii) accelerometer random uncertainty
(viii) accelerometer non-orthogonality
(ix) initial misalignment angles
(x) accelerometer scale factor uncertainty

The input-output results are given on pages 96 - 108 of the detailed report.

March 26, 1979

COMPREHENSIVE MID-TERM REPORT

RESEARCH PROJECT DAERO-77-G-056

<u>GEODETIC APPLICATIONS OF INERTIAL NAVIGATIONAL SYSTEMS</u>

Abstract

A gradiometer-aided inertial navigation system is theoretically and statistically analysed to estimate its abilities to monitore geocentric cartesian coordinates. Having discussed the inertial instrumental units used on the moving platform and several reference coordinate frames applicable in all navigation systems, studies on the severe problem of the separability of the gravity gradients from the inertial disturbances are carried out. Simulation I presents how well the aided navigation system can produce inertial coordinates and how the newcomers of the inertial instrumentation, the gradiometers, perform on-board the moving vehicle. Quantization error studies are also analysed and presented for such a system. Simulation II includes besides the detailed analysis of the accelerometer and gradiometer error models used, the abilities of the system to estimate geocentric coordinates. Multipoint statistical analysis for the approximated inertial acceleration components shows that the navigation system under consideration behaves better as closer the reality is approached.

Zusammenfassung

Ein gradiometer-unterstütztes inertiales Navigationssystem wird theoretisch und statistisch analysiert, um seine Möglichkeiten abzuschätzen, geozentrische cartesische Koordinaten zu ermitteln. Nach einer Diskussion der inertialen Instrumenten-Einheiten auf der bewegten Plattform und einiger Bezugssysteme, die in allen Navigationssystemen angewandt werden, wird das Problem der Trennbarkeit der Schweregradienten von den Inertialstörungen untersucht. Simulation I zeigt, mit welcher Güte das unterstützte Navigationssystem inertiale Koordinaten liefern kann und wie die Neulinge unter den Inertialgeräten, die Gradiometer, sich an Bord des bewegten Fahrzeuges verhalten. Auch der Einfluß von Quantisierungsfehlern wird untersucht und für solch ein System präsentiert. Simulation II enthält neben einer detaillierten Analyse der benutzten Fehlermodelle für Accelerometer und Gradiometer auch die Möglichkeiten des Systems zur Schätzung geozentrischer Koordinaten. Eine statistische Mehrpunkt-Analyse der näherungsweise inertialen Beschleunigungskomponenten zeigt, daß das betrachtete Navigationssystem sich umso besser verhält, je näher man der Realität kommt.

Contents

0. Introduction

Paris, May 1914. Lawrence Sperry is one of the entrants for a prize of fifty thousand francs, the object of the competition being what one judge sees and describes:"the airplane is already in flight and the mechanic rises, leaves his seat and without fear goes from the cabin to the wings and returns back. At the same moment the pilot, Lawrence Sperry, lifts his arms and leaves the airplane to continue its flight without guidance with speed of about 90km/h". He finally won and the first stabilized platform was already introduced for airplane stabilization.

Navigation is perhaps connected with the first human activities on the earth because it is the "art" of obtaining with measurements velocity and position of a moving object. Guidance has to be always distinguished from navigation, for it is the process of generating motion correction commands to a moving object such as it succeeds in its mission. It is understood that guidance includes navigation but not vice-versa.

Guidance techniques could be considered as an extension of human being's natural senses of seeing, smelling, hearing, feeling, memory and deduction. In cases where the guidance problem is not inside the man's abilities, then a "device" could upgrade one or more of his natural senses. For example, man's measuring abilities could be augmented by using electronic measuring devices or computers which can increase his accuracy and speed of deduction. Going a little further where the guidance problem is very complex or the human presence is impossible (e.g. mission leading to destructive termination), then automatic guidance is to be introduced.

Newton's law of mechanical inertia is by far the basic law governing all desired properties of guidance systems. Under this law all particles with mass will exert reaction to the applied acceleration, which is equal in magnitude and opposite in direction, the reaction not being dependent on contacts with the environment.Constructing self-contained instruments dependent upon inertial effects, it is therefore natural to refer to them as inertial guidance systems and in case of navigation as inertial navigation systems.

An inertial navigation system generally contains four basic elements:a)an accelerometer b)an attitude referencec)a computer and d)a clock. An accelerometer is a

device which measures the non-gravitational acceleration experienced upon its case. But, since the case is hard-mounted on the moving vehicle, the measured acceleration is also the acceleration of the vehicle. It is understood that since a vector quantity has three components, then three accelerometers, orthogonally mounted, measure the non-gravitational acceleration resolved in their sensitive or input axes. The orientation of these axes is in a manner best-suited to meet the system's requirements.

The attitude reference is that part of the inertial navigation instrumentation which either stabilizes or commands the platform frame relative to an inertial or rotating frame respectively. Gyroscopes are always called to instrument the attitude reference and considered to be the most indispensable and critical unit on-board.

The computer solves the fundamental equation of inertial navigation to give the velocity and position estimates of the vehicle being navigated. The mathematical procedure to be followed depends on the system requirements as well as the on-board instrumentation. If, for example, the accelerometers used are of the integrated type, then the computer does not perform integrations as far as the accelerometer signal is concerned.

The clock generally gives the time instant of the measurements performed onboard. In cases the navigation takes place relative to an earth rotating frame, then the clock establishes the location or orientation of that frame.

An inertial navigation system may be abstractly considered as a black box. The input to this box is apparent acceleration which contains relative vehicle acceleration, gravitational acceleration and accelerations due to the rotation of the frame of the black box relative to the inertial space. Inside the black box manipulations are carried out and the output is finally the instantaneous velocity and position of the moving vehicle or better of the black box. Needless to say, that the manipulations contain errors and therefore the output is incorrectly indicated. The differences between the actual velocity and position and their output counterparts are then statistically analysed to yield the error budget of the navigation system(black box).

It could be supposed that inertial navigation is accomplished by measuring the apparent acceleration of the moving vehicle and then integrating it

doubly in time to estimate position. But due to Einstein's principle of equivalence, the gravitational acceleration and the non-gravitational one are manifestations of the same basic physical phenomenon. Consequently, gravitational information is certainly needed on-board in order to solve the navigation equation and there is no way out of it.

Tracing the historical developments of inertial navigation, we see how the scientists treated the above postulation due to the lack of actual gravitational information. A reference field was selected, the gravitational acceleration was computed(grossly approximated)at the instantaneous vehicle location and then substructed from the apparent acceleration measurements to go to the solution of the final equation used. Unfortunately, the same procedure is still in use nowadays.

As early as in 1950, Lundberg constructed and tested the first gravity gradiometer. The instrument was composed of two vertically suspended masses and had the ability to sense the sign of the first vertical derivative of gravity. The gradiometer was heavily tested in North America,Europe and West Africa, but nowadays is completely forgotten for not known reasons.

During 1959-62, the Lockwood Company in Toronto introduced its first prototype gradiometer sensing the vertical gradient of gravity with an accuracy of 100Eötvös($1E=10^{-9}sec^{-2}$), but the device was very sensitive to the airborne dynamic environment.

After these first attempts came the era of the first generation of gravity gradiometers. The Hughes, the M.I.T, the Bell Aerospace gradiometers are a sample of very refined and tested gradiometers designed to be used in airborne gradiometry. The feasibility studies have proved so far that an accuracy of 1E or better is to be expected in the very near future and especially for the Hughes gradiometer for which five days of continuous gathering data of the earth's gravity field would be enough to map it completely.

Savet in one of his papers writes:"altogether, it appears that there is no clear-cut advantage in using an existing or feasible gradiometer or, for that matter, a pair of accelerometers"(Savet, 1970). Eight years later, we read something breathtaking:"in the University town of Nancy in France George Delamare declares:<<the key to my operation is a tiny electrode implanted in each of the leg muscles which transmits a computerized electro-

stimulation in response to a peripheral data system based upon micro accelerometers and inclinometers to appreciate the patient(paraplegic)space mission. Experiments carried out in the NASA space programme could be applied to the problem of controlling the balance of the patient when mobile>>"(Time magazine, March 1978).

It seems that sometimes non-geodesists appreciate the geodetic tools better.

In view of the era which the gradiometers promise and the fact that there is not such a gross exaggeration as to assume a spherical earth gravity field(!), we motivate the present analysis to put on-board the moving vehicle a number of gradiometers to measure the earth's gravity field. One could immediately assert that the accuracy of the inertial navigation systems is slightly better than one mile/hour flight and it might be considered as satisfactory. But, again, we do not lay the problem on the desired or obtainable accuracy. Perhaps we do not think so much of the operational point of view, at least, in the very beginning. The navigation systems need, as we believe, a theoretical injection which inevitably comes from the gradiometer implementation.

Consequently, we start the whole analysis fresh from the beginning. We avoid one of the current techniques, that is, first make the assumptions(so bias the system) and then obtain what it might be expected:good results like the assumptions. We go the other way round. Lay the fundaments rigorously and the time for the assumptions will come in order to present some indicated numbers of how well the system behaves. Perhaps, there is a reconciliation between the two approaches.

The newcomers in the inertial navigation instrumentation technology, the gradiometers, promise many applications. As we all know, one of the prime goals in geodesy is the determination of the earth's gravity field. This can be straightforwardly accomplished by employing a moving gravity gradiometer which could furnish even in a very short period the desired gravity data. Other applications of a gradiometer-aided inertial navigation system are:
a)vertical indication
b)vertical deflection indication
c)geoid height indication
just to mention only a few of them. It is therefore concluded that gravity gradiometers offer valuable applications in geodetic science.

The inertial navigation system we analyse is a gradiometer-aided system. Whatever comes as an input into the black box is actually measured on-board. Certain mathematical manipulation takes place inside the box and the output, what we are particularly interested in, is instantaneous vehicle coordinates with respect to a frame rotating with the earth.

Section 1 deals with the inertial navigation instrumental units. Our platform is enriched by three spherical gradiometers developed and tested in the M.I.T. Three gyros command the platform to rotate with the earth's rate and the system has the capability to actually measure and then feed the he on-board computer with all what it needs:apparent acceleration components and the whole gravity gradient tensor.

In section 2, the fundamental equation of inertial navigation for an earth-bound region is derived. Since we are only interested in examining the capabilities of a gradiometer-aided inertial navigation system for terrestrial navigation, the gravitational fields of all attracting masses but the earth's are excluded.

The description of all common coordinate frames used in inertial navigation systems is presented in section 3. As soon as the notion of these frames is completely understood, then one could really have the flexibility to introduce the most general distortions a coordinate frame can undergo. Since our navigation system estimates geocentric coordinates of the mass centre of the platform and it is impossible to "curve in" all on-board frames on that point, due to the actual dimensions of the units, it is assumed that the resulted centrifugal accelerations(each for each instrumental unit mass centre) are negligible small quantities.

Thinking in terms of Einstein's principle of equivalence, it seems to be hopeless to separate gravitational from inertial effects. Studies on this separability are carried out in section 4. It is concluded that gradiometers can measure in a dynamic environment only gravity gradients if and only if they are inertially stabilised. Since our platform is inertially rotating, a second platform is introduced to accommodate only the gradiometer measurement unit stabilized with respect to an inertial frame.

An extensive literature exists on simulation studies of inertial navigation systems. Depending on the system used and the assumptions set, different results have been drawn so far. The performance of the inertial instrumentation

is considered,nowadays, to be satisfactory, but the systems still gather an appreciable error budget. Updating the system from time to time , the errors are reduced but they are too far from being eliminated. Since there is not any actual in flight gradiometer, it is assumed and sometimes believed that the approximation of the earth's gravity field has the worst contribution into the system's accuracy. For that reason, we try to identify with a simple example the function which drives the errors in an inertial navigation system. Section 5 contains a simulation to this direction not including error models for the instrumental units. Finally, a space traverse is computed to see the behaviour of such an aided system.

Section 6 examines the quantization of errors for the accelerometers and gradiometers. Assuming that each time the instruments are read a quantization error is present, a general formula is given to estimate the position error due to the instrumental truncations treated as stochastic quantities.

A general error model is analysed in section 7. We go to the far end of each measurement unit taking into account what they actually measure and considering the possible types on instrumental misalignment and non-orthogonality. The errors which contribute more than 90% of the whole error budget are paid attention and included in the general model. On the final navigation equation suitably approximated, the second simulation studies are performed to test the capabilities of a gradiometer-aided inertial navigation system.

1. Inertial measurement units

1.1 Accelerometers

1.1.1 General considerations

Just a glance at the fundamental equation of inertial navigation(eq.(2.11)) shows how critical is the acceleration contribution to the navigation problem. As a matter of fact, inertial navigation can be principally accomplished by measuring only the apparent acceleration on-board a moving vehicle. The gravity field compensation could be computed by employing a reference field, such as an ellipsoidal one, and using approximate position values, so as the gravity components at the instrumentation location are then to be estimated.

The number of the on-board accelerometers depends on the particular problem considered. Generally speaking, when we navigate in the three dimensional space, three accelerometers are used in order to sense all three apparent acceleration components. In case of a cruise aircraft, the vertical accelerometer could be substituted by a barometer or an altimeter reducing the number to two.

The accelerometers provide their measurements in a frame which they are constrained to follow and it is furnished by the gyros. In our analysis, three accelerometers are set at the platform's mass centre such as to construct an orthogonal frame and measure the apparent acceleration vector resolved in their rotating(with earth's rate) axes.

1.1.2 Operational principles

Let us now see how an ideal accelerometer operates and in order to motivate our discussion, consider an ideal spring-mass accelerometer shown in Fig.(1). The instrument consists of:

a) the case
b) the proof mass and
c) the damping spring

As the accelerometer follows the motion of the vehicle, the apparent acceleration acts on the proof mass as well as the case. The proof mass extends

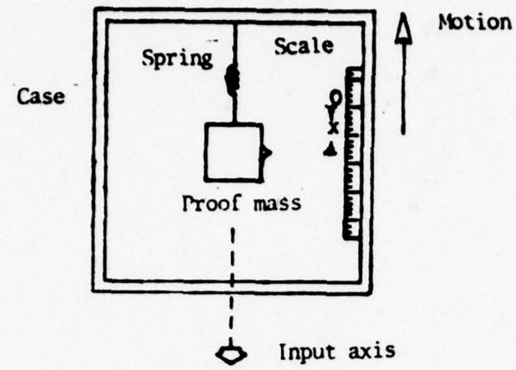


Fig.(1): An ideal spring-mass accelerometer

the spring and the displacement x is measured against a properly divided scale. Generally, the equation governing the spring-mass accelerometer operation reads

(1.1.1) $A = m\ddot{x}+d\dot{x}+kx$

where A the apparent acceleration, x the scale reading, m the proof mass, d the damping factor and k the spring constant. Having got the x scale reading and known the mentioned constants, the apparent acceleration can be then computed using eq.(1.1.1). Needless to say, that the apparent acceleration is sensed along the input accelerometer axis and therefore three such accelerometers orthogonally mounted can fully estimate the apparent acceleration vector.

Finally, we remark that an ideal accelerometer is nothing else but an ideal gravimeter and since a moving gravimeter cannot discriminate between gravitational and non-gravitational forces, the accelerometer output is a mixture of these two force fields and as such it must be gravimetrically compensated.

1.1.3 Error model

As we said before, three accelerometers can measure the apparent acceleration vector and their output contributes to the estimation of the inertially referenced acceleration. But, since it is impossible to construct an orthogonal accelerometer frame, the apparent acceleration is sensed along a quasi-orthogonal frame and care should be taken to compensate for accelerometer non-

orthogonality. The transformation of the apparent acceleration vector from the quasi-orthogonal to the ideal orthogonal accelerometer frame is a small angle transformation and it is considered in full detail in section 7.1. Furthermore, each accelerometer has its own bias and scale factor uncertainty which falsify the sensed acceleration. Taking all these errors into account, the accelerometer signal is corrected and refers to the ideal accelerometer frame ready to be transformed to any desired coordinate frame in which we like to solve the fundamental equation of inertial navigation.

1.2 Gradiometers

1.2.1 General considerations

Right from the beginning of the application of the inertial navigation systems, it was fully understood that a serious contributing factor in their error budget was the gravity field compensation in the accelerometers measurements. At that time, there was no space for envisioning an instrument mounted on-board a moving vehicle and having the capability of directly measuring the gravity field. As the years went by and the inertial instrumentation and guidance systems reached such a tremendous qualification and performance, it was quite evident that a research for the first generation of moving gravity instrumentation was inevitable. So far, the earth's gravity field was approximated by the gravity field of an ellipsoid of revolution. Given approximate position values, the gravity field is computed at the vehicle's instantaneous location using well-known closed formulas. Having granted by the gradiometer instrumentation, we feel a little disturbed to continue approximating the reality as far as the gravity field is concerned. Of course, the motivation for studying and constructing gradiometers was not, at least primarily, to aid inertial navigation systems. But since gravity is a critical inertial parameter, we think that it is really a good chance to have on-board a gradiometer, or a number of them, in an effort to actually measure the gravity field in which the navigation takes place.

Nowadays, the gradiometers are still in the feasibility study phase except some on-board tests(see P.Hood and H.Ward,1969,p.98), but their performance promises very quick on-board implementation(Trageser,1975). Consequently, it is reasonable to assume that under the assertions of the mechanical

engineers, gradiometer applicability is warranted in a serious effort to aid autonomous inertial navigation systems.

1.2.2 Operational principles

A gradiometer measures the changes of the gravity components with respect to the displacement,that is,gravity gradients. The advantages of a gradiometer arereally a lot. Moving base capability, no Eötvös correction, no terrain correction, just to mention a few of them. Among the most interesting ones are:

a)the Bell Aerospace gradiometer

b)the Hughes gradiometer and

c)the M.I.T. gradiometer

For a description see(Williams,1975). In 1915, the intelligency of Rolan von Eötvös created the torsion balance bearing his name and measuring certain gravity gradients. Without exception, its principles apply to the up-to-date gradiometers. Consider two equal proof masses connected with an axle and supported at a flexure point between them. When that primitive instrument passes over a mass anomaly, different(in the differential sense) attraction forces are exerted upon the proof masses due to the difference in the distance between them and the mass anomaly. Consequently, the instrument changes its position about the flexure point and if this change(rotation) is appropriately sensed, then gravity gradients could be measured. Of course, the measurement of a gradient is very complicated due to the advanced electronics involved.

In what follows, we shall try to explain briefly the spherical gradiometer developed in the Massachusetts Institute of Technology(M.I.T.) which is used as our on-board gradiometer. Since all the known gradiometers are under laboratory tests,there is not any immediate justification why we prefer that instrument to the others. But we believe, at least from the given accuracies (Trageser,1975, Forward, 1974, Williams, 1975), that the M.I.T. spherical gradiometer possesses certain advantages in comparison with the others e.g. structural stability, immunity to platform jitter rectification effects etc.

Let us therefore study the operational principles of the spherical gradiometer. The instrument consists of (see Fig.(2)):

a) the float and

b) the housing of the float.

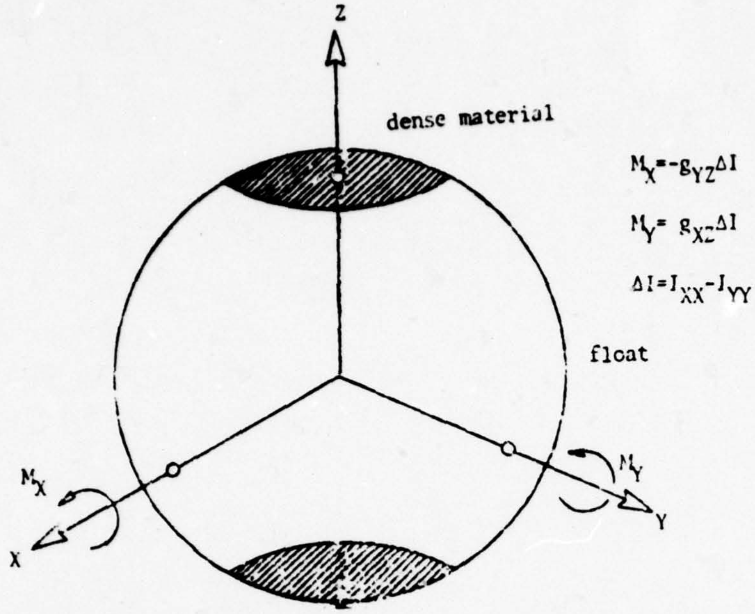


Fig.(2): Spherical gradiometer float

Between the float and the housing, a small gap exists which is filled by a special fluid supporting the float inside the stabilized housing. When the instrument passes over a mass anomaly, the float is constrained to rotate inside the housing. The moment about a defined axis is measured by the electronics of the housing and is applied back as a restoration torque in order to bring the float to its original position. For example, the moment measured about the X axis (Fig.(2)) gives the g_{YZ} gravity gradient and the moment about the Y axis the g_{XZ} one. Therefore, each instrument measures two gradients(another great advantage) and three of them mounted on a platform can provide us the whole gravity gradient tensor.

Feasibility studies on the spherical gradiometer support the expectation that 1E or better could be attained in the near future(Trageser, 1975)

1.2.3 Error model

In trying to construct a general gradiometer error model, three reference frames are employed in conjuction with the gradiometer float:

a) the electronic frame, which senses the small float rotations(tôrques)

b) the float or misaligned gradiometer measurement unit frame, which is fixed with respect to the float principal moments of inertia axes and

c) the ideal gradiometer measurement unit frame to which the gravity gradients refer.

In a general error model, the first two reference frames should be considered as a quasi-orthogonal ones and the gravity gradients signal must be accordingly corrected using the involved misalignment angles. Furthermore, gradiometers are instruments which inertially measure gravity gradients. Since our gradiometer measurement unit frame is designed to have the same orientation as an earth centered quasi-inertial frame, a general misalignment of the former frame with respect to the latter one should be taken into account. When all these misalignments induced gravity gradients errors are considered, then the gravity gradients signal can be transformed to the desired coordinate frame in which we like to solve the fundamental equation of inertial navigation.

1.3. Gyroscopes

1.3.1. General considerations

It is well-known that inertial navigation reaches its aim in determining velocity and position with respect to a reference frame by the implementation of three mutually perpendicular mounted accelerometers which measure the components of the apparent acceleration vector resolved in their input or sensitive axes. It is noting by passing, that these measurements must be gravitationally compensated. Let us now suppose that we like to measure the apparent acceleration in a moving vehicle in order to determine, say, position. For that purpose, three accelerometers are mounted on-board the moving vehicle having their sensitive axes to a known orientation relative to a reference frame with respect to which we navigate. As soon as the vehicle starts moving and begins pitching, rolling and yawing, then it is absolutely impossible for the accelerometer axes to preserve their original orientation. Consequently, their output cannot be used as an input for solving the fundamental equation of inertial navigation. A special device must be implemented in order to "dump out" the time-like accelerometer frame misorientations. This device should have the capability of either to command the accelerometer frame(or for that matter the platform frame) to rotate,in case the navigation frame rotates or to stabilize it to a desired orientation, in case of an inertially non-rotating reference frame. This can be accomplished, as we will see, by employing three single-degree-of freedom gyros or two two-degree-of freedom which can cover the three possible degrees of

freedom in space.

In what follows, we shall try to give briefly the gyro functional characteristics and their error model which is included in the simulation studies.

1.3.2 Operational principles

Let us suppose that a solid body rotates with angular velocity ω along an axis of symmetry (Fig.(3a)). Newton's second law in rotational form states that in an inertial frame of reference:

$$\vec{T} = \frac{d\vec{H}}{dt} \qquad (1.3.1)$$

$$\vec{H}_i = I_{ij}\vec{\omega}_j$$

where $\vec{T}$ represents the applied torque, $\vec{H}$ the angular momentum and I the moment of inertia of the rotating solid mass. Now, let mount this simple device on a case with a single axis gimbal as in (Fig.(3b)). This over-simplified mechanization depicts the principle of the single-degree-of-freedom gyro (SDF) owing its name to the single gimbal suspension. A SDF gyro is composed of:

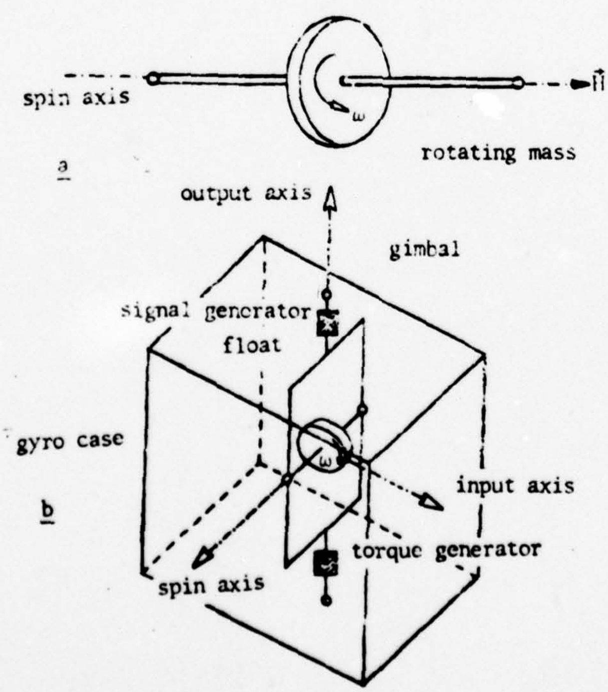


Fig.(3): Single-degree-of-freedom gyro.

a) the gyro case or housing

b) the gyro float and

c) the gyro rotor

From eq.(1.3.1), it is understood that in absence of applied torques, its integration gives

$$\vec{H} = \text{constant}$$

and this manifests the most fundamental property of a gyro:when no external torque is present, then the direction of the angular momentum vector preserves its orientation in space.

In order to practically utilize the property of the constancy of the angular momentum orientation, the H vector ought to be in the direction of one axis of the rotating body symmetry. If it is not the case, then the angular momentum could be resolved in three components as:

$$\begin{aligned} &\vec{H}_s = \vec{I}_s \cdot \vec{\omega}_s \\ (1.3.2)\quad &\vec{H}_{t_1} = \vec{I}_t \cdot \vec{\omega}_1 \\ &\vec{H}_{t_2} = \vec{I}_t \cdot \vec{\omega}_2 \end{aligned}$$

where $\vec{\omega}_s$ the rotation along the axis of symmetry and $\vec{\omega}_1, \vec{\omega}_2$ the rotations along two(any) perpendicular axes. Denoting the angle between $\vec{H}$ and the axis of symmetry by $\vec{\psi}$, we note that this is a constant angle, since it depends on the I's and ω's which are constant quantities for a solid body rotating with constant speed. But due to the principle that the angular momentum vector preserves its orientation in space, then it is concluded that the axis of symmetry must move otherwise it would coincide with $\vec{H}$. Consequently, the axis of symmetry generates a cone around H with apex angle 2ψ and we refer to it as the (free)gyroscopic nutation phenomenon.

Suppose now that a torque is applied perpendicular to the angular momentum vector H. In this case, H rotates with angular velocity $\vec{\theta}$ transverse to both torque and $\vec{H}$ vectors and it is such that the applied torque is equal to the cross product of $\vec{H}$ and the rotation vector $\vec{\theta}$. This rotation of the angular momentum vector, being influenced by torques, is called precession. In case in which the torques come from the friction of the gimbal bearings, then the angular momentum vector again precesses and thus its initial orientation is continuously changing. In this case, we speak of gyro drift which is one of the most critical errors in inertial navigation.

Let us now examine how three gyros can either stabilize or command a moving platform to a desired rate. Consider the SDF-gyro of (Fig.(3b)) mounted on a platform which generally rotates as the vehicle moves. Along the output axis of the gyro two special devices are set, the signal generator(SG) and the torque generator(TG). The former has the ability to sense torques due to the rotation of the gimbal and to apply them as restoration torques. The latter can generate torques according to the rate of rotation of the navigation frame. For example, if we navigate relative to an earth rotating frame, then the torque generator is mechanized to provide torques of, approximately, 15deg/h and the signal generator senses the rotation of its gimbal due to the irregular motion of the vehicle and restores it in order to isolate the platform from being affected by vehicle manoeuvres. In case we navigate relative to an inertially non-rotating reference frame, then only the signal generator operates provided that the gyro frame has already been initially aligned.

Since in the 3D-space, we have three degrees of freedom, it is understood that three gimbals provide complete isolation of the platform's rotation with respect to the turbulent motion of the moving object. Consequently, when we speak of three on-board gyros,we mean that three gimbals preserve the gyro frame to change its attitude as the vehicle is pitching, rolling and yawing.

1.3.3 Error model

Various gyro error models have been suggested in an effort to include all possible error sources in a gyro performance(e.g. Britting, 1971, p.74). The use of a specific model depends on the particular problem which one faces. As we have already explained, our gyros command the platform to the earth's rate and thus, the error model should be selected such as to include the instrumental misfunctions in generating the earth's rotation.

Let us suppose that we have on-board the moving vehicle three SDF-gyros which materilize the gyro frame generating the earth' s rate. Since it is technically impossible to construct a pure orthogonal gyro frame, then the resulting frame is a *non-orthogonal or quasi-orthogonal frame*. Consequently, the torques are applied through a non-orthogonal frame and this should be taken into account. Furthermore, the torques are experienced through the torque generator which have scale factor uncertainty. Therefore, the signal for the earth's rotation comes erroneous and should anyway be adjusted be-

fore it is applied, taking into consideration the two error sources just mentioned. The analysis on the gyro error model is given in section 7.1.

2. The fundamental equation of inertial navigation

By the term fundamental equation of inertial navigation, we mean that equation whose solution estimates the running values of the moving object. In our analysis, geocentric coordinates are the desired output of its solution. Consider the one dimensional navigation example of a train. Its fundamental navigation equation is distance equals velocity (considered as constant) multiplied by time. Measuring time by an on-board clock, we can estimate the instantaneous position of the train with respect to a convenient initial point.

The final expression of the fundamental equation depends on the coordinate frame in which it is coordinatized as well as the motion of the platform frame relative to the navigation frame used. For example, if the navigation takes place in an inertial coordinate frame and the platform is inertially stabilized, then the navigation equation assumes its simplest form. In case in which the platform is constrained to rotate, then the navigation equation includes more terms such as accelerations of Coriolis and centripetal type.

We shall try to derive the fundamental equation of inertial navigation in the general case in which the navigation frame rotates inertially and the platform frame does also the same. Then, it is easy to specialize it to any desired simplification. We finally remark that the navigation equation is valid in any coordinate system, since it is nothing else than a vector equation. It comes straightforward from the total derivative of a vector quantity. Consider two coordinate frames, a fixed one E_1 and a moving E_2. The transformation between them can be represented

$$\vec{E}_2 = R\,\vec{E}_1$$

Differentiating, we get

$$d\vec{E}_2 = dR\,\vec{E}_1 + R\,d\vec{E}_1 = dR\,\vec{E}_1 = dR\,R^T\,\vec{E}_2 = \Omega\,\vec{E}_2$$

since $d\vec{E}_1 = 0$ being fixed and Ω is known as the Cartan matrix. Now, consider a vector which can be expressed in the two coordinate frames E_1 and E_2 as

$$\vec{A} = a_i \vec{E}_1^i = b_i\,\vec{E}_2^i$$

where a_i and b_i the coordinates of the vector A and $a_i\vec{E}_1^i$, $b_i\vec{E}_2^i$ its compo-

nents in E_1 and E_2 frames respectively. Differentiating again we get:

$$\begin{aligned} &d\vec{A} = da_i\vec{E}_1^i + a_i d\vec{E}_1^i = db_i\,\vec{E}_2^i + b_i\,d\vec{E}_2^i \qquad \text{or} \\ (2.1)\quad & \\ &d\vec{A} = da_i\vec{E}_1^i = db_i\,\vec{E}_2^i + b_i\,\Omega\,\vec{E}_2^i \end{aligned}$$

Eq.(2.1) can be interpreted in the following way. The derivative of a vector with respect to an inertial frame is equal to its derivative with respect to a moving frame(such as one fixed at the earth's centre) plus the rotation of the moving frame with respect to the inertial one multiplied by the vector itself. The last term is called the velocity of following. We state again this important conclusion using notation valid in what follows as:

$$(2.2)\qquad \frac{d_N\vec{r}}{dt} + \Omega_N^I\vec{r} = \frac{d_I\vec{r}}{dt}$$

where I stands for the inertial frame and N for the navigation one. Now let us derive the fundamental equation of inertial navigation using eq.(2.2). Consider a moving vehicle O connected by a vector $\vec{R}$ to the navigation frame N and by r to the inertial frame I (Fig.4). We remind again that $\vec{R}$ terminates at the platform's mass centre and when we speak of a moving vehicle, we mean only a point, its platform's mass centre. From the vectors definition, we get

$$(2.3)\qquad \vec{r} = \vec{R} + \vec{\rho}$$

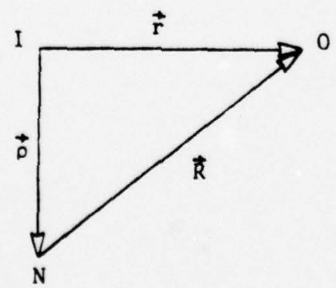


Fig.(4): Inertial navigation geometry.

where $\vec{\rho}$ the separation of the origins of the inertial and navigation frames. Differentiating with respect to the inertial frame we get:

(2.4) $$\dot{\vec{R}}_I + \dot{\vec{\rho}}_I = \dot{\vec{R}}_N + \dot{\vec{\rho}}_I + \vec{\Omega}^I_N \wedge \vec{R}_N = \frac{d_I}{dt}\vec{r}_I$$

(2.5) $$\ddot{\vec{R}}_N + \vec{\Omega}^I_N \wedge \dot{\vec{R}} = \frac{d_I}{dt}\dot{\vec{R}}_N$$

(2.6) $$(\dot{\vec{\Omega}}^I_N)_I \wedge \vec{R} + \vec{\Omega}^I_N \wedge \dot{\vec{R}}_I = (\dot{\vec{\Omega}}^I_N)_I \wedge \vec{R} + \vec{\Omega}^I_N \wedge (\dot{\vec{R}}_N + \vec{\Omega}^I_N \wedge \vec{R}) = \frac{d_I}{dt}(\vec{\Omega}^I_N \wedge \vec{R})$$

(2.7) $$\ddot{\vec{r}}_I = \ddot{\vec{\rho}}_I + \ddot{\vec{R}}_N + \vec{\Omega}^I_N \wedge \dot{\vec{R}}_N + \dot{\vec{\Omega}}^I_N \wedge \vec{R} + \vec{\Omega}^I_N \wedge (\dot{\vec{R}}_N + \vec{\Omega}^I_N \wedge \vec{R})$$

Eq.(2.7) expresses the inertial acceleration of the platform's mass centre as a function of the acceleration and velocity of the vehicle with respect to the inertial one. For further reference, eq (2.7) is written in the form:

(2.8) $$\ddot{\vec{r}}_I = \ddot{\vec{\rho}}_I + \ddot{\vec{R}}_N + 2\vec{\Omega}^I_N \wedge \dot{\vec{R}}_N + \dot{\vec{\Omega}}^I_N \wedge \vec{R} + \vec{\Omega}^I_N \wedge (\vec{\Omega}^I_N \wedge \vec{R})$$

It is worth noting that the inertial acceleration is the sum of the gravitational and non-gravitational acceleration. Consequently

(2.9) $$\ddot{\vec{r}}_I = \vec{A} + \vec{G}$$

and eq.(2.8) assumes now the form

(2.10) $$\ddot{\vec{R}}_N = \vec{A} + \vec{G} - (\ddot{\vec{\rho}}_I + 2\vec{\Omega}^I_N \wedge \dot{\vec{R}}_N + \dot{\vec{\Omega}}^I_N + \vec{\Omega}^I_N \wedge (\vec{\Omega}^I_N \wedge \vec{R}_N))$$

If we want to estimate geocentric coordinates, in case in which the navigation frame is linked to the earth, then we have to integrate eq.(2.10) doubly with respect to time. But,as one could foresee, there exists some difficulties in performing the integration, namely:

a) the angular velocity of the navigation frame reffered to the inertial space must be known. Consequently, it is intuitively understood that the navigation frame should be linked to a body, with respect to which we like to navigate, which has known angular motion characteristics with respect to the inertial frame of reference. Linking the navigation frame to the earth's centre, then its inertially referenced angular velocity is a priori known quantity and it is undoubtedly a very good choice. As a matter of

fact, this is common practice in inertial navigation, when the navigation takes place near the earth's space.

b) the inertially referenced angular acceleration of the navigation frame must be known. But, in view of the obtained accuracies of the up-to-date navigation systems, this quantity is very small and could be neglected.

c) the acceleration of the navigation frame with respect to the inertial one is involved in eq.(2.10). This really a problematic quantity as far as its measurements can be accomplished. To motivate our discussion, consider the centre of mass of the universe. This point satisfies the requirement for constant speed rectilinear motion, that is, it is an inertial point. Now, centering three axes at that point with known space directions, we get an inertial frame of reference. But how to measure the acceleration of the navigation frame fixed, say, at the earth's centre of mass with respect to the inertial frame? And even if we continuously approximate the inertial frame with other quasi-inertial ones coming closer and closer to the navigation frame, certain difficulties in measuring the inertial acceleration exist again. To overcome this cumbersome situation, let us divide the universe into two regions, an internal region and an external one. Thus, the gravitational acceleration acting on the moving object is the sum of the gravitational acceleration of the internal and external regions or

$$\vec{G} = \vec{G}_{cxt} + \vec{G}_{int}$$

If we consider now that the external gravitational field G_{ext} is uniform, that is to say, it has the same direction and magnitude everywhere in the external region, then every body moving in the external region will "get" the same gravitational acceleration coming from the external region. Since in this region the non-gravitational forces on the moving body are zero, then we can state

$$\vec{G}_{ext} = \ddot{\vec{\rho}}_I$$

In view of this result, eq.(2.10) reads

$$\ddot{\vec{R}}_N = \vec{A} + \vec{G}_{int} - (2\vec{\Omega}_N^I \wedge \dot{\vec{R}}_N + \dot{\vec{\Omega}}_N^I \wedge \vec{R} + \vec{\Omega}_N^I \wedge (\vec{\Omega}_N^I \wedge \vec{R}))$$

Now, a very reasonable question arises:with what criteria one could divide the universe into these two regions? where are the boundaries of them? The answer becomes simple in case we know in what kind of navigation we are involved. In our studies, the navigation takes place near the earth's sur-

face, that is to say, we are interested in terrestrial navigation. Consequently, the boundary between the external and internal regions is some kilometers above the earth's surface and as such the gravitational field of the internal region represents only the earth's gravity field. The next to the mentioned one strong gravity fields, namely the moon's and sun's fields, are neglected being quantities up to the order of 10^{-7} the earth's gravity field intensity. Finally, the fundamental equation of inertial navigation, in case of terrestrial navigation, reads:

$$\ddot{\vec{R}}_E = \vec{A} + \vec{G}_E - 2\vec{\Omega}^I_E \wedge \dot{\vec{R}}_E - \dot{\vec{\Omega}}^I_E \wedge \vec{R} - \vec{\Omega}^I_E \wedge (\vec{\Omega}^I_E \wedge \vec{R}) \tag{2.11}$$

where $\vec{A}$: the apparent acceleration of the moving vehicle

$\vec{G}_E$: the earth's gravity field

$2\vec{\Omega}^I_E \wedge \dot{\vec{R}}_E$: the Coriolis acceleration

$\vec{\Omega}^I_E \wedge (\vec{\Omega}^I_E \wedge \vec{R})$: the centripetal acceleration

and the navigation frame is centered at the earth's centre of mass.

Eq.(2.11) stands for what we are after. It represents the fundamental equation of inertial (terrestrial) navigation being investigated to visualize the capabilities of a gradiometer-aided inertial navigation system to estimate instantaneous geocentric coordinates of a moving vehicle inside the earth's space.

It is practically worthwhile to give some indicating numbers concerning the magnitude of the Coriolis and centripetal acceleration. Consider Ω in eq.(2.11) to represent the earth's angular velocity and an aircraft which moves with velocity of 500 km/h. Then, the Coriolis acceleration accounts for

$$2\Omega^I_E \cdot \dot{R}_E = 2\ (7{,}29)(10^{-5})\text{rad/sec}\ 500\text{km/h} \approx 4 \cdot 10^{-3} g$$

where g the earth's gravity field intensity. It is understood, that the Coriolis acceleration magnitude depends upon the velocity of the moving vehicle. The centripetal acceleration for an earth bound region accounts for almost the same magnitude given above.

3. Reference frames used in inertial navigation

The discussion about any inertial navigation configuration intuitively includes the introduction of some coordinate frames which are either instrumented on-board the moving vehicle or used as reference frames and as such the system output is related to them. The inclusion and orientation of these frames depend upon the particular purpose of the navigation system, but there are some of them which have to be anyway implemented, since they play fundamental role in the whole mathematical analysis.

The coordinate frames used in inertial navigation could be classified into two distinct categories:

3.1:the external coordinate frames,which are linked to bodies other than the moving vehicle and

3.2:the internal or on-board coordinate frames, which are linked to the object being navigated.

In the first category two particular classes can be defined:

3.1.1:the inertial frame,where the term stands for a number of quasi-inertial frames approximating the absolute one. Concerning the accuracy of the up-to-date inertial navigation systems, it is not reasonable to desperately include the use of the one real inertial system. Any of the first approximations can be undoubtedly considered as satisfactory. One could adopt a reference system with origin at the earth's centre of mass(included its atmosphere) and with the following axes orientation:

3-axis: towards the instantaneous rotation axis of the earth
1-axis: towards the vernal equinox
2-axis: completes the right-handed orthogonal system

Such a quasi-inertial frame is a common choice in inertial navigation applications, since its angular velocity with respect to the fixed stars is a negligible small quantity, if it is compared to the short period of operation of a navigation system. The frame discussed plays the role of the inertial frame in the present work, the inertial property being understood in the given reasoning.

3.1.2 : the navigation frame.It is defined as that frame in which the final output of the navigation system is coordinatized. A lot of choices could bring a lot of navigation frames into picture, the particular choice depented upon the purpose of the navigation system under consideration. In our

analysis, we like to estimate geocentric coordinates and therefore the navigation frame must be linked to the earth. The orientation of that frame is as follows:

3-axis: towards the rotation axis of the earth considered as a solid body. The axis is fixed at the time instant at which the navigation starts. If we like to refer to the instantaneous rotation axis of the earth, then polar motion must be taken into account.

1-axis: towards the intersection of the equator fixed when the navigation begins and the Greenwich meridian defined by a set of world-wide ,well-distributed astronomical stations.

2-axis: completes the right-handed orthogonal frame

The origin of the navigation frame is at the earth's mass centre included its atmosphere.

Now, let us discuss the internal or on-board frames. The description includes all or at least most of the on-board frames, but this does not mean that they have to be anyway implemented.

3.2.1 <u>the computation frame</u>.It exists only in the on-board computer memory and it serves as that frame in which the final manipulation of the navigation system equations is carried out. Having decided to estimate geocentric coordinates, then the computation frame is set to the same orientation as the navigation frame and remains parallel to it everafter.

3.2.2 <u>the mechanized or ideal platform frame</u>.It is the frame with respect to which the gyro and accelerometer frames are held constant. As the word "ideal" explains, that frame is practically distorted due to various instrumental reasons e.g. gyro drift, initial misalignment etc. The ideal platform frame, in some applications, may differ from the navigation frame, the motivation for that being what we like to take out of the system. In our analysis, the ideal platform frame is identical with the computation frame, because there is no reason to assume the opposite.

3.2.3 <u>the actual platform frame</u>. The initial platform misalignment causes the first effect and as the time passes the gyro drift drives the ideal platform frame away from its desired orientation. The resulting frame is the actual platform one. The drifting platform affects, finally, the orientation of the gyro and accelerometer frames which lose their ability to stabilize the platform and sense correctly the apparent acceleration components respectively.

3.2.4 <u>the gyro frame</u>.It is by far the most important configuration in any navigation system. As a matter of fact, without gyros there is no way to navigate a moving vehicle(at least with the up-to-date technology). The three orthogonal output axes of the on-board gyros construct the so-called gyro frame which possesses the fundamental principle to preserve its orientation regardless the turbulent motion of the carrier. Consequently, every distortion of the platform's desired orientation can be sensed by the gyros and applied as restoration torque to correct the platform to its proper attitude. Unfortunately, the gyro frame drifts with time and this complicates the navigation analysis.

3.2.5 <u>the accelerometer frame</u>.The three input axes of the on-board accelerometers instrument the accelerometer frame along which the apparent acceleration is sensed. The frame is affected by its non-orthogonality as well as the drifting platform.

3.2.6 <u>the gradiometer frame</u>.As in the gyros' and accelerometers' case, the axes of the three gradiometers construct the gradiometer frame. It is worth noting, that this frame comes into picture only when one likes to actually measure gravity gradients. Besides, gradiometers measure pure gravity gradients(without inertial disturbances) if and only if they follow an inertial orientation. Consequently, the gradiometer frame must be inertially stabilized.

Regarding the definitions of the above frames, the following comments might offer clarity and understanding in what follows:

a) The aim of our navigation system is to compute the instantaneous geocentric coordinates of the moving vehicle. In order to be exact, we must declare that when we speak of a moving vehicle , it must be understood as only the centre of mass of the moving platform. All the mathematical relationships as well as the estimated geocentric coordinates refer to this point. But, there are two arguments against this:

I: The on-board instrumentation cannot, of course, be concentrated on the the mass centre of the platform and it is actually distributed around this point in a radious of a few meters.

II: The gradiometers are tremendously sensitive instruments. Thus, it is strongly recommended that they ought to operate in the tail of an aircraft, in case of aircraft navigation, so as to avoid influence of the movements of the personnel etc.

The above two arguments cause no trouble because the centrifugal accelera-

tions created from the distribution of the instrumentation around the platform's centre of mass (which should be otherwise included) are now dropped of the navigation system's equations as negligible small quantities.

b) It is, finally, worth repeating that the orientation of the navigation, computation and ideal platform frame is the same. This gives simplicity, to some extent, on the derivation of the system's navigation equation and is by no means an assumption.

4. Separability of gravity gradients and inertial disturbances

The subject of this analysis is of principal importance in case in which gradiometers are used on-board a moving vehicle to measure gravity gradients. The central question arisen is whether we can measure pure gravity gradients while we are moving or not. In other words, we have to find out what is the output of a gradiometer being affected by the turbulent motion of the carrier.

It is more than seventy years that all of us have been benefited by the Einstein's principle of equivalence according to which the acceleration field is equivalent to the gravitational field. It is worth noting here, that this principle holds only approximately and locally and only in that sense we cannot separate gravitational from acceleration effects (Fock,1964). Let us therefore examine in what mode a gradiometer measures the gravity gradients of the gravitational field in which it operates.

Consider again two frames, an inertial one denoted by X and an accelerated with respect to the inertial denoted by x. The relationship between them reads

$$X_i = A_{ij}x_j + D_i$$

where A_{ij} represents the relative orientation of the two frames and D_i the displacement of the origin of the moving frame with respect to the inertial origin. Performing in exactly the same mode as in the derivation of the fundamental equation of inertial navigation and representing the force per unit mass by F, then we get

$$(4.1)\qquad \frac{d^2x_k}{dt^2} = A_{ik}F_i - 2A_{ik}\frac{dA_{ij}}{dt}\frac{dx_j}{dt} - A_{ik}\frac{d^2A_{ij}}{dt^2}x_j - A_{ik}\frac{d^2D_i}{dt^2}$$

Let us now determine the terms involved in eq.(4.1). Using the Kronecker delta definition for the rotation matrix we obtain

$$A_{ik}\cdot A_{ij} = \delta_{kj}$$

and by differentiation

$$(4.2)\qquad \frac{dA_{ik}}{dt}A_{ij} + A_{ik}\frac{dA_{ij}}{dt} = 0$$

or

$$(4.3)\qquad \Omega_{kj}+\Omega_{jk} = 0$$

with the obvious substitution representing the instantaneous inertially referenced angular velocity of the moving frame. Differentiating Ω_{jk} with respect to time we get

$$(4.4)\qquad \frac{d\Omega_{jk}}{dt} = \frac{dA_{ik}}{dt}\frac{dA_{ij}}{dt} + A_{ik}\frac{d^2A_{ij}}{dt^2}$$

and according to eqs(4.2) and (4.3), we write

$$(4.5)\qquad \Omega_{ij}\Omega_{kj} = \frac{dA_{ij}}{dt}\frac{dA_{ik}}{dt}$$

Consequently, eq.(4.1) is now written as

$$(4.6a)\qquad \frac{d^2x_i}{dt^2} = f_i + 2\Omega_{ij}\frac{dx_j}{dt} + x_i\left(\frac{d\Omega_{ij}}{dt} + \Omega_{ik}\Omega_{jk}\right) - \frac{d^2D_i}{dt^2}$$

where f_i represents the coordinates of the gravity vector. Eq.(4.6a) does not assume the Newtonian form and if it is stretched to do so, we obtain

$$(4.6b)\qquad \frac{d^2x_i}{dt^2} = A_i = f_i + x_j\left(\frac{d\Omega_{ij}}{dt} + \Omega_{ik}\Omega_{jk}\right) - \frac{d^2D_i}{dt^2}$$

where the term dx_j/dt has been dropped out, since the moving instruments are mounted on the platform and therefore, there is no relative motion. Eq.(4.6b) is solved with respect to $f_i = \partial V/\partial x_i$ and then, successive space differentiation yields the gravity gradients and the third gravitational derivatives respectively. Consequently,

$$(4.7)\qquad \frac{\partial V_i}{\partial x_i} = A_i - x_j\left(\frac{\partial\Omega_{ij}}{\partial t} + \Omega_{ik}\Omega_{jk}\right) - \frac{d^2D_i}{dt^2}$$

$$(4.8)\qquad \frac{\partial^2 V}{\partial x_i\partial x_j} = \frac{\partial A_i}{\partial x_j} - \left(\frac{\partial\Omega_{ij}}{\partial t} + \Omega_{ik}\Omega_{jk}\right)$$

$$(4.9)\qquad \frac{\partial^3 V}{\partial x_i\partial x_j\partial x_k} = \frac{\partial^2 A_i}{\partial x_j\partial x_k}$$

From eqs(4.7),(4.8) and(4.9) the following conclusions can be drawn:

a) The structural difference between the gravitational and acceleration field allows the third derivative of the gravitational field to be free from inertial disturbances. If an instrument could be designed to measure the third derivatives, then its measurements would be pure quantities of

the gravitational field, the motion of the vehicle having no effect on them whatsoever.

b) The gravity components are always affected by inertial disturbances. This explains the fact why a gravimeter cannot perform gravity measurements while it is moving.

c) When the platform rotates and translates in space, then gravity gradients are mixed with inertial disturbances. The way out is to inertially stabilize the gradiometer frame and then the second term in eq.(4.8) representing the inertial disturbances is cancelled. But as we have designed our platform, it trucks an earth fixed geocentric frame and therefore, it rotates with earth's rate. To reconcile the constraints posed on the gradiometers, a second platform is set on the moving vehicle which is inertially non-rotating and on this platform the gradiometers perform the pure gravity gradients measurements.

The above discussion was motivated by the following two very important reasons:

1) If one is keen on asking what a gradiometer measures, then eqs(4.7), (4.8) and (4.9) give the straightforward answer. It measures what one likes as well as what one dislikes. It is in one's choice,merely, to exclude inertial effects by posing the constraints which come so clearly from these equations.

2) In our work, we like to estimate geocentric coordinates referred to an earth-fixed, non-inertial frame. But our discussion has proved that gradiometers must measure only relative to an inertial frame and as soon as pure gravity gradients have been obtained, then they can be transformed to any other reference frame such as, say, to the earth-fixed one. Only under these lines, it is well understood that the separability of gravitational and inertial effects is of great importance, when gradiometers are called to measure only pure gravity gradients on-board a moving vehicle.

5. The system's driving error function

5.1 General considerations

It is well-known that inertial navigation systems are burdened with a large ammount of errors which, in some cases, turns on to be intolerable. For example, the vertical channel of an autonomous navigation system is so unstable that the system is practically out of its reasonable operation as far as this channel is concerned, after a short time of operation. The general instability of an autonomous inertial navigation system can be faced:

a) with an external aid: various instruments have been suggested and used so far in an effort to furnish additional on-board measurements with the aim of reducing a specific channel or a whole navigation system's instability. For the vertical channel a barometer or an altimeter could accomplish it quite succesfully (Winter, 1974). For a whole system, a velocitymeter or a laser equipment which measures distances or a gradiometer or a camera taking photographs while it is moving, are some of the external aids which can effectively reduce the navigation system's errors down to very reasonable values.

b) with a periodic updating: in many inertial navigation applications, the system becomes rather quickly very unstable due to the presence of big errors. The remedy for that is the application of certain mathematical methods,i.e. Kalman filtering, to update the system at a desired observation point and to "start" it fresh henceforth. It is understood, that the system's updating takes place rather often, its periodicity being depended upon how quick the errors propagate, the sampling interval etc.

Our anlysis deals with the first method described above. So far the earth's gravity field is approximated by the gravity field of a sphere or an ellipsoid of revolution. This is, of course, only an approximation of the reality which in its turn contributes errors to the navigation system performance. In each science special assumptions are always set in an effort to approach the reality as closer as possible. But, in view of the hopeful results coming from the gradiometer feasibility studies and tests, we could take full advantage of them. Now, we have an instrument which has the ability to measure the gravity field. Thus, why to make such a vital assumption? Finally, we enrich our platform instrumentation by setting a number of gradiometers

to measure the earth's gravity field. Certain assumptions have of cource to be made concerning gradiometer accuracy, values of the gravity gradients in space, location of the platform's instrumentation etc.

5.2 The simulated navigation equation

Our study examines one of the simple cases of an inertial navigation system, trying to find the function which drives the error budget of the system under consideration. In order to bring the fundamental equation of inertial navigation into an easily simulated form, certain assumptions have to be made, e.g. absence of instrumental errors. We believe that making these assumptions, we do not overshadow problems, but we present a simplified example to draw a very interesting conclusion, namely, which is the worst contributed factor in an inertial navigation system.

Let us, now, examine in detail our navigation system. The moving inertial platform is on-board an aircraft and is composed of the following instrumentation:

a) three orthogonally mounted single-axis accelerometers in order to measure the components of the apparent acceleration vector resolved into their axes.
b) three orthogonally mounted gyroscopes which can isolate the platform from the turbulent motion of the aircraft and preserve the orientation of the platform frames with respect to the inertial frame.
c) three orthogonally mounted spherical gradiometers which furnish the measurements of the entire gravity gradient tensor.

Taking into account that the moving platform does not rotate with respect to the inertial frame, which is in this case the navigation frame, then all terms in the general navigation equation(eq.(2.11)) containing the angular velocity term, are simply dropped out. Consequently, the navigation equation covering our system assumes the simple form

$$\ddot{R}_I = A + G_I \tag{5.2.1}$$

where R denotes the instantaneous inertial position of the aircraft, subscript I shows the inertial reference, A is the apparent acceleration and G the gravity components. For simplicity, the inertial subscript is dropped henceforth. Eq.(5.2.1) can be written in component form as

$$\ddot{X}=A_X+G_X$$

$$\ddot{Y}=A_Y+G_Y \tag{5.2.2}$$

$$\ddot{Z}=A_Z+G_Z$$

where X,Y and Z the inertial coordinates of the aircraft. Since we do not measure gravity components but gravity gradients and the inertial acceleration terms call for undesired integration, eqs(5.2.2) are approximated in the following reasoning:

a) the inertial acceleration components could be substituted by the inertial coordinates, using the well-known Stirling's approximation formula applied for three successive points(Scheid, 1968)

$$\ddot{X}_i=\frac{X_{i+1}-2X_i+X_{i-1}}{\Delta t^2} \tag{5.2.3}$$

b) the gravity components are to be approximated by the gravity gradients using a coordinate-free derivation which is valid(as the name explains) in any coordinate frame:

$$G_i = G_{i-1}+\mathrm{grad}G_{i-1}(R_i-R_{i-1})+ \text{ higher order terms} \tag{5.2.4}$$

where the higher order terms are neglected.

Consequently, eq.(5.2.2) in view of the approximations made could be rewritten in the form:

$$(X_{i+1}-2X_i+X_{i-1})\Delta t^{-2}=A_{X_i}+G_{X_i}+G_{XX_{i-1}}(X_i-X_{i-1})+G_{XY_{i-1}}(Y_i-Y_{i-1})+G_{XZ_{i-1}}(Z_i-Z_{i-1})+$$

quantization error

$$(Y_{i+1}-2Y_i+Y_{i-1})\Delta t^{-2}=A_{Y_i}+G_{Y_i}+G_{YX_{i-1}}(X_i-X_{i-1})+G_{YY_{i-1}}(Y_i-Y_{i-1})+G_{YZ_{i-1}}(Z_i-Z_{i-1})+ \tag{5.2.5}$$

quantization error

$$(Z_{i+1}-2Z_i+Z_{i-1})\Delta t^{-2}=A_{Z_i}+G_{Z_i}+G_{ZX_{i-1}}(X_i-X_{i-1})+G_{ZY_{i-1}}(Y_i-Y_{i-1})+G_{ZZ_{i-1}}(Z_i-Z_{i-1})+$$

quantization error

Quantization error studies are performed in section 6. Eqs(5.2.6) show a rather peculiar phenomenon. When the moving platform is at the point i

$$X_{i+1} = \Delta t^2(A_{X_i}+G_{X_i}+G_{XX_{i-1}}(X_i-X_{i-1})+G_{XY_{i-1}}(Y_i-Y_{i-1})+G_{XZ_{i-1}}(Z_i-Z_{i-1}))+2X_i-X_{i-1}+ \text{ q.e.}$$

$$Y_{i+1} = \Delta t^2(A_{Y_i}+G_{Y_i}+G_{YX_{i-1}}(X_i-X_{i-1})+G_{YY_{i-1}}(Y_i-Y_{i-1})+G_{YZ_{i-1}}(Z_i-Z_{i-1}))+2Y_i-Y_{I-1}+ \text{ q.e.} \tag{5.2.6}$$

$$Z_{i+1} = \Delta t^2(A_{Z_i}+G_{Z_i}+G_{ZX_{i-1}}(X_i-X_{i-1})+G_{ZY_{i-1}}(Y_i-Y_{i-1})+G_{ZZ_{i-1}}(Z_i-Z_{i-1}))+2Z_i-Z_{i-1}+ \text{q.e.}$$

and the on-board instrumentation measures apparent acceleration and gravity gradients(as a matter of fact the measurements are performed between the points (i-1) and i), then the system estimates the inertial coordinates of the (i+1)-point. Therefore, the platform behaves like a "moving window" and there are two general ways one could face this situation:

a) the Boundary Value Problem of Inertial Navigation:given the inertial coordinates of the first and the final points, then the system can estimate all other points, but only off-flight, that is, post mission. In some applications, this has certain advantages. For example, consider a photogrammetric airplane taking photographs of an area. During the flight, the instantaneous coordinates of the moving vehicle are not of direct interest. But, after the mission has been accomplished, the desired photo-manipulation needs the coordinates of the camera from where all photographs have been taken. Thus, application of eqs(5.2.6) simultaneously for all in flight points gives the required values.

b) the Initial Value Problem of Inertial Navigation:when the first two points of the navigation path have known coordinates, then eqs(5.2.6) estimate the inertial coordinates of all other points to come in flight. In many practical inertial navigation applications, as in terrestrial navigation, we are most interested in knowing at any time instant where we are and therefore this case suits to our analysis. Schematically, our navigation problem is illustrated in Fig.(5).

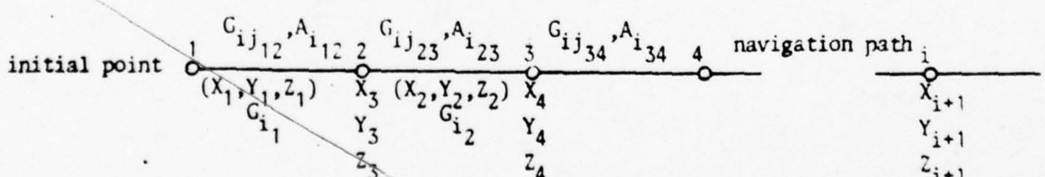


Fig.(5): The initial value problem of inertial navigation.

In Fig.(5), quantities above the navigation path are to be measured and under it are known. The quantities inside the brackets under the navigation path indicate what is estimated when the platform is at that point.

Before we compute the matrices involved in the variance-covariance expressions for the estimated inertial coordinates, it is worthwhile to examine the impact of the Laplace condition on the variances of the measured gra-

vity gradients. It is well-known that the in-line gravity gradients are connected with the following condition:

$$G_{ZZ} = -G_{XX} - G_{YY} \tag{5.2.7}$$

known as the Laplace condition. For instructive purposes, we rewrite the gravity gradient tensor

$$\begin{bmatrix} G_{XX} & G_{XY} & G_{XZ} \\ G_{YX} & G_{YY} & G_{YZ} \\ G_{ZX} & G_{ZY} & G_{ZZ} \end{bmatrix}$$

and note the solenoidal and irrotational structure of the gravity field ($G_{ij}=G_{ji}$). What is above the dotted line is observed and certain accuracies are later to be assumed for them. But, the vertical in-line gravity gradient (G_{ZZ}) which enters eq.(5.2.7) has to get a variance dependent on the other two gravity gradients variances. Consequently, trying to find the relationship between the in-line gravity gradients variances, we write

$$G_{XX} = G_{XX} + 0G_{YY}$$

$$G_{YY} = 0G_{XX} + G_{YY}$$

$$G_{ZZ} = -G_{XX} - G_{YY}$$

or in matrix form

$$\begin{bmatrix} G_{XX} \\ G_{YY} \\ G_{ZZ} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} G_{XX} \\ G_{YY} \end{bmatrix} \tag{5.2.8}$$

If eq.(5.2.8) is abstractly written as Y=AX and the law of propagation of errors is applied, then we obtain

$$\begin{bmatrix} \sigma^2_{G_{XX}} & \sigma_{G_{XX}G_{YY}} & \sigma_{G_{XZ}G_{ZZ}} \\ \sigma_{G_{YY}G_{XX}} & \sigma^2_{G_{YY}} & \sigma_{G_{YY}G_{ZZ}} \\ \sigma_{G_{ZZ}G_{XX}} & \sigma_{G_{ZZ}G_{YY}} & \sigma^2_{G_{ZZ}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \sigma^2_{G_{XX}} & \sigma_{G_{XX}G_{YY}} \\ \sigma_{G_{YY}G_{XX}} & \sigma^2_{G_{YY}} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

$$
(5.2.9) \quad = \begin{bmatrix} \sigma^2_{G_{XX}} & \sigma_{G_{XX}G_{YY}} & -\sigma^2_{G_{XX}}-\sigma_{G_{XX}G_{YY}} \\ \sigma_{G_{XX}G_{YY}} & \sigma^2_{G_{YY}} & -\sigma_{G_{XX}G_{YY}}-\sigma^2_{G_{YY}} \\ -\sigma^2_{G_{XX}}-\sigma_{G_{YY}G_{XX}} & -\sigma_{G_{XX}G_{YY}}-\sigma^2_{G_{YY}} & \sigma^2_{G_{XX}}+\sigma^2_{G_{YY}}+\sigma_{G_{XX}G_{YY}} \end{bmatrix}
$$

Equating the respective matrix elements, we get besides the obvious identities the following interesting expressions:

$$
\begin{aligned}
&\sigma_{G_{ZZ}G_{XX}} = -\sigma^2_{G_{XX}} - \sigma_{G_{XX}G_{YY}} \\
(5.2.10) \quad &\sigma_{G_{ZZ}G_{YY}} = -\sigma^2_{G_{YY}} - \sigma_{G_{XX}G_{YY}} \\
&\sigma^2_{G_{ZZ}} = \sigma^2_{G_{XX}} + \sigma^2_{G_{YY}} + 2\sigma_{G_{XX}G_{YY}}
\end{aligned}
$$

The above equations are of value for our analysis because
a)when we consider the observables to be uncorrelated, then we cannot assume the same for the vertical gravity gradient and thus its correlation with respect to the other two in-line gradients ought to be considered.
b)when we simulate the navigation system, certain variances have to be chosen for the G_{XX}, G_{YY} gradients. G_{ZZ} will then get a variance as eq. (5.2.10c) indicates.

What we intend to examine now is the contribution of each observable of the navigation system to its error budget. The simulation takes place when the platform is at the 2-point and so, the equations can analytically be presented as:

$$
\begin{aligned}
&X_3 = \Delta t^2 (A_{X_{12}} + G_{X_1} + G_{XX_{12}}(X_2-X_1) + G_{XY_{12}}(Y_2-Y_1) + G_{XZ_{12}}(Z_2-Z_1)) + 2X_2 - X_1 \\
(5.2.11) \quad &Y_3 = \Delta t^2 (A_{Y_{12}} + G_{Y_1} + G_{YX_{12}}(X_2-X_1) + G_{YY_{12}}(Y_2-Y_1) + G_{YZ_{12}}(Z_2-Z_1)) + 2Y_2 - Y_1 \\
&Z_3 = \Delta t^2 (A_{Z_{12}} + G_{Z_1} + G_{ZX_{12}}(X_2-X_1) + G_{ZY_{12}}(Y_2-Y_1) + G_{ZZ_{12}}(Z_2-Z_1)) + 2Z_2 - Z_1
\end{aligned}
$$

In the system of eqs(5.2.11) we have 19 parameters, namely

a) Δt:the time interval between two successive observations(1)

b) A_i:the components of the apparent acceleration(3)

c) G_i:the gravity components(3)

d) G_{ij}:three in-line and three off-line gravity gradients(6)

e) $(X,Y,Z)_1,(X,Y,Z)_2$:the inertial coordinates of the first two points(6)

Generally, the variance-covariance matrix of the estimated inertial coordinates can be written

$$\Sigma_Y = G^T \Sigma_X G \tag{5.2.12}$$

where Σ_Y indicates the variance-covariance matrix of the coordinates, Σ_X the variance-covariance matrix of the 19 parameters, G the matrix of the derivatives of eqs(5.2.11)with respect to the 19 parameters and T denotes the transpose matrix. In view of eqs(5.2.11), the simulated system is given in table (1). The derivatives of eqs(5.2.11) with respect to the parameters involved read:

$$\frac{\partial X_3}{\partial \Delta t} = 2\Delta t(A_{X_{12}}+G_{X_1}+G_{XX_{12}}(X_2-X_1)+G_{XY_{12}}(Y_2-Y_1)+G_{XZ_{12}}(Z_2-Z_1))$$

$$\frac{\partial X_3}{\partial A_{X_{12}}} = \Delta t^2$$

$$\frac{\partial X_3}{\partial G_{X_1}} = \Delta t^2$$

$$\frac{\partial X_3}{\partial G_{XX_{12}}} = \Delta t^2(X_2-X_1)$$

$$\frac{\partial X_3}{\partial G_{XY_{12}}} = \Delta t^2(Y_2-Y_1)$$

$$\frac{\partial X_3}{\partial G_{XZ_{12}}} = \Delta t^2(Z_2-Z_1)$$

$$\frac{\partial X_3}{\partial X_1} = -(1+\Delta t^2 G_{XX_{12}})$$

$$\frac{\partial X_3}{\partial Y_1} = -\Delta t^2 G_{XY_{12}}$$

$$\begin{bmatrix} \sigma^2_{X_3} & \sigma_{X_3Y_3} & \sigma_{X_3Z_3} \\ \sigma_{Y_3X_3} & \sigma^2_{Y_3} & \sigma_{Y_3Z_3} \\ \sigma_{Z_3X_3} & \sigma_{Z_3Y_3} & \sigma^2_{Z_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial X_3}{\partial \Delta t} & \frac{\partial X_3}{\partial A_{X_1}} & 0 & 0 & \frac{\partial X_3}{\partial G_{X_1}} & 0 & 0 & \frac{\partial X_3}{\partial G_{XX_1}} & \frac{\partial X_3}{\partial G_{XY_1}} & \frac{\partial X_3}{\partial G_{XZ_1}} & 0 & 0 & 0 & \frac{\partial X_3}{\partial X_1} & \frac{\partial X_3}{\partial Y_1} & \frac{\partial X_3}{\partial Z_1} & \frac{\partial X_3}{\partial X_2} & \frac{\partial X_3}{\partial Y_2} & \frac{\partial X_3}{\partial Z_2} \\ \frac{\partial Y_3}{\partial \Delta t} & 0 & \frac{\partial Y_3}{\partial A_{Y_1}} & 0 & 0 & \frac{\partial Y_3}{\partial G_{Y_1}} & 0 & 0 & \frac{\partial Y_3}{\partial G_{YX_1}} & 0 & \frac{\partial Y_3}{\partial G_{YY_1}} & \frac{\partial Y_3}{\partial G_{YZ_1}} & 0 & \frac{\partial Y_3}{\partial X_1} & \frac{\partial Y_3}{\partial Y_1} & \frac{\partial Y_3}{\partial Z_1} & \frac{\partial Y_3}{\partial X_2} & \frac{\partial Y_3}{\partial Y_2} & \frac{\partial Y_3}{\partial Z_2} \\ \frac{\partial Z_3}{\partial \Delta t} & 0 & 0 & \frac{\partial Z_3}{\partial A_{Z_1}} & 0 & 0 & \frac{\partial Z_3}{\partial G_{Z_1}} & 0 & 0 & \frac{\partial Z_3}{\partial G_{ZX_1}} & 0 & \frac{\partial Z_3}{\partial G_{ZY_1}} & \frac{\partial Z_3}{\partial G_{ZZ_1}} & \frac{\partial Z_3}{\partial X_1} & \frac{\partial Z_3}{\partial Y_1} & \frac{\partial Z_3}{\partial Z_1} & \frac{\partial Z_3}{\partial X_2} & \frac{\partial Z_3}{\partial Y_2} & \frac{\partial Z_3}{\partial Z_2} \end{bmatrix}$$

$$
\left[\begin{array}{ccccccccccccccccccc}
\sigma^2_{\Delta t} \\
 & \sigma^2_{A_{X_1}} \\
 & & \sigma^2_{A_{Y_1}} & & & & & & & & & & & & 0 \\
 & & & \sigma^2_{A_{Z_1}} \\
 & & & & \sigma^2_{G_{X_1}} \\
 & & & & & \sigma^2_{G_{Y_1}} \\
 & & & & & & \sigma^2_{G_{Z_1}} \\
 & & & & & & & \sigma^2_{G_{XX_1}} \\
 & & & & & & & & \sigma^2_{G_{XY_1}} \\
 & & & & & & & & & \sigma^2_{G_{XZ_1}} \\
 & & & & & & & & & & \sigma^2_{G_{YY_1}} \\
 & & & & & & & & & & & \sigma^2_{G_{YZ_1}} \\
 & & & & & & & \sigma_{G_{XX}G_{ZZ}} & & & \sigma_{G_{YY}G_{ZZ}} & & \sigma^2_{G_{ZZ_1}} \\
 & & & & & & & & & & & & & \sigma^2_{X_1} \\
 & & & & & & & & & & & & & & \sigma^2_{Y_1} \\
 & & 0 & & & & & & & & & & & & & \sigma^2_{Z_1} \\
 & & & & & & & & & & & & & & & & \sigma^2_{X_2} \\
 & & & & & & & & & & & & & & & & & \sigma^2_{Y_2} \\
 & & & & & & & & & & & & & & & & & & \sigma^2_{Z_2}
\end{array}\right]
\left[\begin{array}{c|c|c}
\partial X_3/\partial \Delta t & \partial Y_3/\partial \Delta t & \partial Z_3/\partial \Delta t \\
\hline
\partial X_3/\partial A_{X_1} & 0 & 0 \\
\hline
0 & \partial Y_3/\partial A_{Y_1} & 0 \\
\hline
0 & 0 & \partial Z_3/\partial A_{Z_1} \\
\hline
\partial X_3/\partial G_{X_1} & 0 & 0 \\
\hline
0 & \partial Y_3/\partial G_{Y_1} & 0 \\
\hline
0 & 0 & \partial Z_3/\partial G_{Z_1} \\
\hline
\partial X_3/\partial G_{XX_1} & 0 & 0 \\
\hline
\partial X_3/\partial G_{XY_1} & \partial Y_3/\partial G_{YX_1} & 0 \\
\hline
\partial X_3/\partial G_{XZ_1} & 0 & \partial Z_3/\partial G_{ZX_1} \\
\hline
0 & \partial Y_3/\partial G_{YY_1} & 0 \\
\hline
0 & 0 & \partial Z_3/\partial G_{ZY_1} \\
\hline
0 & 0 & \partial Z_3/\partial G_{ZZ_1} \\
\hline
\partial X_3/\partial X_1 & \partial Y_3/\partial X_1 & \partial Z_3/\partial X_1 \\
\hline
\partial X_3/\partial Y_1 & \partial Y_3/\partial Y_1 & \partial Z_3/\partial Y_1 \\
\hline
\partial X_3/\partial Z_1 & \partial Y_3/\partial Z_1 & \partial Z_3/\partial Z_1 \\
\hline
\partial X_3/\partial X_2 & \partial Y_3/\partial X_2 & \partial Z_3/\partial X_2 \\
\hline
\partial X_3/\partial Y_2 & \partial Y_3/\partial Y_2 & \partial Z_3/\partial Y_2 \\
\hline
\partial X_3/\partial Z_2 & \partial Y_3/\partial Z_2 & \partial Z_3/\partial Z_2
\end{array}\right]
\qquad (5.2.13)
$$

Table (1): The simulated navigation equation.

$$\frac{\partial X_3}{\partial Z_1} = -\Delta t^2 G_{XZ_{12}}$$

$$\frac{\partial X_3}{\partial X_2} = \Delta t^2 G_{XX_{12}} + 2$$

$$\frac{\partial X_3}{\partial Y_2} = \Delta t^2 G_{XY_{12}}$$

$$\frac{\partial X_3}{\partial Z_2} = \Delta t^2 G_{XZ_{12}}$$

$$\frac{\partial Y_3}{\partial \Delta t} = 2\Delta t(A_{Y_{12}} + G_{Y_1} + G_{YX_{12}}(X_2 - X_1) + G_{YY_{12}}(Y_2 - Y_1) + G_{YZ_{12}}(Z_2 - Z_1))$$

$$\frac{\partial Y_3}{\partial A_{Y_{12}}} = \Delta t^2$$

$$\frac{\partial Y_3}{\partial G_{Y_1}} = \Delta t^2$$

$$\frac{\partial Y_3}{\partial G_{YX_{12}}} = \Delta t^2 (X_2 - X_1)$$

$$\frac{\partial Y_3}{\partial G_{YY_{12}}} = \Delta t^2 (Y_2 - Y_1)$$

$$\frac{\partial Y_3}{\partial G_{YZ_{12}}} = \Delta t^2 (Z_2 - Z_1) \tag{5.2.14}$$

$$\frac{\partial Y_3}{\partial X_1} = -\Delta t^2 G_{YX_{12}}$$

$$\frac{\partial Y_3}{\partial Y_1} = -(1 + \Delta t^2 G_{YY_{12}})$$

$$\frac{\partial Y_3}{\partial Z_1} = -\Delta t^2 G_{YZ_{12}}$$

$$\frac{\partial Y_3}{\partial X_2} = \Delta t^2 G_{YX_{12}}$$

$$\frac{\partial Y_3}{\partial Y_2} = \Delta t^2 G_{YY_{12}} + 2$$

$$\frac{\partial Y_3}{\partial Z_2} = \Delta t^2 G_{YZ_{12}}$$

$$\frac{\partial Z_3}{\partial \Delta t}=\Delta t^2(A_{Z_{12}}+G_{Z_1}+G_{ZX_{12}}(X_2-X_1)+G_{ZY_{12}}(Y_2-Y_1)+G_{ZZ_{12}}(Z_2-Z_1))$$

$$\frac{\partial Z_3}{\partial A_{Z_{12}}}=\Delta t^2$$

$$\frac{\partial Z_3}{\partial G_{Z_1}}=\Delta t^2$$

$$\frac{\partial Z_3}{\partial G_{ZX_{12}}}=\Delta t^2(X_2-X_1)$$

$$\frac{\partial Z_3}{\partial G_{ZY_{12}}}=\Delta t^2(Y_2-Y_1)$$

$$\frac{\partial Z_3}{\partial G_{ZZ_{12}}}=\Delta t^2(Z_2-Z_1)$$

$$\frac{\partial Z_3}{\partial X_1}=-\Delta t^2 G_{ZX_{12}}$$

$$\frac{\partial Z_3}{\partial Y_1}=-\Delta t^2 G_{ZY_{12}}$$

$$\frac{\partial Z_3}{\partial Z_1}=-(1+\Delta t^2 G_{ZZ_{12}})$$

$$\frac{\partial Z_3}{\partial X_2}=\Delta t^2 G_{ZX_{12}}$$

$$\frac{\partial Z_3}{\partial Y_2}=\Delta t^2 G_{ZY_{12}}$$

$$\frac{\partial Z_3}{\partial Z_2}=\Delta t^2 G_{ZZ_{12}}+2$$

5.3 Results

Eq.(5.2.13) is our simulated navigation equation combined with eqs(5.2.14). In order to obtain the desired results, certain accuracies have been assumed for the 19 parameters of the system. The simulation covers the case in which the observables are not correlated. When an observable assumes a ran-

ge of values, the rest 18 are held constant so as the influence of the observable under consideration on the variances of the inertial coordinates could be deduced. The used ground values for the 19 parameters of the system are listed in Table(2).

Parameter	Value	Variance
Δt	0.1sec	0.001sec^2
A_{X_1}	0.1m/sec^2	0.01m^2/sec^4
A_{Y_1}	0.1	0.01
A_{Z_1}	0.1	0.01
G_{X_1}	0.5	0.01
G_{Y_1}	0.5	0.01
G_{Z_1}	9.0	0.01
G_{XX_1}	-1500.E	1.E
G_{XY_1}	-0.2	1.
G_{XZ_1}	-15.	1.
G_{YY_1}	-1500.	1.
G_{YZ_1}	+0.05	1.
G_{ZZ_1}	+3000.	1.
X_1	6200000 m	1.m^2
Y_1	6000000	1.
Z_1	6500000	1.
X_2	6200030	1.
Y_2	6000030	1.
Z_2	6500030	1.

Table (2): Ground values for the parameters used in the simulation.

As far as these values are concerned the following remarks are indispensable:

a) the **operation of the** platform's instrumentation every Δt=0.1sec. could be generally considered as rapid, but actual tests on aided inertial navigation systems recommend that value (Denhard, 1977).

b) the values for the initial inertial coordinates are selected to be outside the earth's surface having 30m distance apart. An aircraft velocity of 1000km/h is assumed for the moving platform. But, generally speaking, the given coordinate values have not any special meaning whatsoever.

c) the ground values for the gravity gradients have been taken from (Trageser, 1971) and compared to those given by (Reed, 1973). It is worth noting here that for our simulation studies only indicated values for the gravity gradients fulfill the requirements of the analysis, since we do not derive coordinates, but the influence of each observable on the variances of the coordinates.

For the computational analysis, a special programme was written in the sense of eqs(5.2.12) and compiled in the Siemens Computer at the University FAF Munich. The results of the carried out simulation are given below.

$\sigma^2_{G_X}$ = (m^2/sec^4)	var.-cov.:		(m^2)
0.02	5.0340	0.0040	0.0400
		5.0240	0.0400
			5.4201
0.05	5.0640	0.0040	0.0400
		5.0240	0.0400
			5.4201
0.10	5.1140	0.0040	0.0400
		5.0240	0.0400
			5.4201
0.20	5.2104	0.0040	0.0400
		5.0240	0.0400
			5.4201
0.50	5.5140	0.0040	0.0400
		5.0240	0.0400
			5.4201

$\sigma^2_{G_Z}$ = (m^2/sec^4)	var.-cov.:		(m^2)
0.02	5.0240	0.0040	0.0400
		5.0240	0.0400
			5.4301
0.05	5.0240	0.0040	0.0400
		5.0240	0.0400
			5.4601
0.10	5.0240	0.0040	0.0400
		5.0240	0.0400
			5.5101
0.20	5.0240	0.0040	0.0400
		5.0240	0.0400
			5.6101
0.50	5.0240	0.0040	0.0400
		5.0240	0.0400
			5.9101

$\sigma^2_{X_1}=$ (m^2)	var.-cov.:		(m^2)
	4.0340	0.0040	0.0400
0.01		5.0240	0.0400
			5.4201
	4.0740	0.0040	0.0400
0.05		5.0240	0.0400
			5.4201
	4.1240	0.0040	0.0400
0.10		5.0240	0.0400
			5.4201
	4.5240	0.0040	0.0400
0.50		5.0240	0.0400
			5.4201
	6.0240	0.0040	0.0400
2.00		5.0240	0.0400
			5.4201
	9.0240	0.0040	0.0400
5.00		5.0240	0.0400
			5.4201

$\sigma^2_{X_2}=$ (m^2)	var.-cov.:		(m^2)
	1.0640	0.0040	0.0400
0.01		5.0240	0.0400
			5.4201
	1.2240	0.0040	0.0400
0.05		5.0240	0.0400
			5.4201
	1.4240	0.0040	0.0400
0.10		5.0240	0.0400
			5.4201
	3.0240	0.0040	0.0400
0.50		5.0240	0.0400
			5.4201
	9.0240	0.0040	0.0400
2.00		5.0240	0.0400
			5.4201
	21.0240	0.0040	0.0400
5.00		5.0240	0.0400
			5.4201

$\sigma^2_{G_{XX}}=$ (E)	var.-cov.:		(m^2)
	5.0241	0.0040	0.0400
10		5.0241	0.0400
			5.4202
	5.0240	0.0040	0.0400
50		5.0245	0.0402
			5.4207
	5.0240	0.0040	0.0400
100		5.0250	0.0405
			5.4213
	5.0240	0.0040	0.0400
1000		5.0340	0.0450
			5.4326

$\sigma^2_{\Delta t}=$ (sec^2)	var.-cov:		(m^2)
	5.0240	0.0040	0.0400
0.001		5.0240	0.0400
			5.4201
	5.2200	0.2000	1.9998
0.05		5.2200	1.9998
			25.0213
	5.4199	0.3999	3.9995
0.1		5.4200	3.9998
			45.0225
	7.0194	1.9996	19.9976
0.5		7.0197	19.9991
			205.0321

$\sigma^2_{Z_1} =$ (m^2)	var.-cov.:		(m^2)
	5.0240	0.0040	0.0400
0.01		5.0240	0.0400
			4.4301
	5.0240	0.0040	0.0400
0.05		5.0240	0.0400
			4.4701
	5.0240	0.0040	0.0400
0.10		5.0240	0.0400
			4.5201
	5.0240	0.0040	0.0400
0.50		5.0240	0.0400
			4.9201
	5.0240	0.0040	0.0400
2.00		5.0240	0.0400
			6.4201
	5.0240	0.0040	0.0400
5.00		5.0240	0.0400
			9.4201

$\sigma^2_{Z_2} =$ (m^2)	var.-cov.:		(m^2)
	5.0240	0.0040	0.0400
0.01		5.0240	0.0400
			1.4601
	5.0240	0.0040	0.0400
0.05		5.0240	0.0400
			1.6201
	5.0240	0.0040	0.0400
0.10		5.0240	0.0400
			1.8201
	5.0240	0.0040	0.0400
0.50		5.0240	0.0400
			3.4201
	5.0240	0.0040	0.0400
2.00		5.0240	0.0400
			9.4201
	5.0240	0.0040	0.0400
		5.0240	0.0400
			21.4201

$\sigma^2_{A_X} =$ (m^2/sec^4)	var.-cov.:		(m^2)
	5.0211	0.0020	0.0204
0.01		5.0240	0.0400
			5.4201
	5.0211	0.0021	0.0208
0.02		5.0240	0.0400
			5.4201
	5.0212	0.0022	0.0220
0.05		5.0240	0.0400
			5.4201
	5.0215	0.0024	0.0240
0.10		5.0240	0.0400
			5.4201

$\sigma^2_{X_i, Y_i, Z_{i=1,2}} =$ (m^2)	var.-cov.:		(m^2)
	0.0740	0.0040	0.0400
0.01		0.0740	0.0400
			0.4701
	0.5240	0.0040	0.0400
0.10		0.5240	0.0400
			0.9201
	2.5240	0.0040	0.0400
0.50		2.5240	0.0400
			2.9201
	10.0240	0.0040	244.0042
2.00		10.0240	244.0225
			4882.5818

From the obtained values the following conclusions can be drawn:
a)as the accuracy of the clock-timer, which signals the time interval Δt and alerts the platform instrumentation to operate, decreases, then the variance of the Z-coordinate gets very big values. Generally speaking, it is reasonable to assume that the clock-timer possesses no error in determining Δt(observe also the increasing values of the covariances!).
b)the accuracy with which the inertial coordinates of the second point are known causes worse instability than that of the first point. The variance of the X-coordinate has no effect on the derived Z-coordinate variance and the deviations in Z cause bigger instability to Z-variance than that due to X or Y variations.
c)the accuracy of the gravity gradients has almost no effect on the estimation of the inertial coordinates. This is really something surprising and not known so far. It can be explained as follows:if one performs the tedious manipulation on the simulated navigation eq.(5.2.13), one could see that all terms containing gravity gradients are multiplied by the fourth power of the time interval Δt and by other small quantities. Thus, their influence is strongly reduced.
d)the variance of the gravity components of the first point has almost no influence on the variance of the coordinates of the third point. Even if the variance is equal to the value of the gravity component(!), then the variance of the third point is burdened by half a meter more.
e)acceleration seems to have no great influence on the derived accuracy of the third point. But, as we shall see later, acceleration variation creates the biggest errors in the system among the 19 observables.

So far the analysis has given the first results, namely, the behaviour of the navigation system with respect to the parameters. Next we try to investigate how the system performs going from point to point. During that process the observables are undergone small changes except for the variances of the coordinates of the first two points which are assumed to be zero. The same computer procedure is used and applied for some first points due to reasons which will be explained later on. The space traverse gives the following results:(see next page)

5.4 Discussion

First, it has to be noted that the statistical analysis up to now has not

Point	σ_X^2	(m^2) σ_Y^2	σ_Z^2
3	0.000016	0.000016	0.003314
4	0.000080	0.000080	0.016570
5	0.000352	0.000352	0.072908
6	0.001504	0.001504	0.3115
7	0.006384	0.006384	1.3222
8	0.02706	0.02706	5.6036
9	0.1146	0.1146	23.7399
10	0.4855	0.4855	100.5665
11	2.0566	2.0566	426.0092
12	8.7095	8.7095	1804.6066
13	36.8966	36.8966	7644.4123

Table(3): A space traverse of a gradiometer-aided inertial navigation system.

included any error model of the platform instrumentation. But, certain errors do exist which sometimes turn to be important. Gyro drift, initial platform misalignment, gyro non-orthogonality etc. are some errors which in a rigorous statistical analysis have to be modelled and taken into consideration. All these problems are thoroughly examined in simulation II.

The performed simulation studies did have the objective to be simple and as such the following conclusions can be drawn:

a)first of all it is clear that the gradiometer-aided inertial navigation system we analyse does not perform well as far as the accuracy of the derived coordinates is concerned. The first few points, say up to the tenth, could be reached with satisfactory precision(see Table(3)). Then the accumulation of the system's errors becomes so high that the performance of it can be considered as unreasonable.

b)manipulating the expressions of the variances-covariances of the simulated equations(Table(1)), we find that besides all other terms, there are two of great interest

1)the variance of the (i-1)-point multiplied by four and

2)the variance of the (i-2)-point

in case the inertial instrumentation is at the i-point. It is worth noting that both terms have positive sign.

c)the answer to the bad behaviour of the system lies on the conclusion b). If we assume that the first two sets of the inertial coordinates have zero variances, then all other terms of the variance-covariance expressions are summed to give the variances of the coordinates of the third point. Four times this variance plus the variance of the second point (plus other small terms) will give us the variance for the fourth point. The same procedure is applied through all other points. It is therefore clearly understood that the following approximate law

i-point variance = 4·(i-1)-point variance+(i-2)-point variance

is valid for the space traverse under consideration. Consequently, an appreciable percentage of the navigation system's error budget comes from the approximation used.

d)next we try to see the error contribution of the acceleration and gravity gradients measurements into the system. To that objective two additional space traverses are performed. In the first one all other quantities except the apparent acceleration components assume zero variances throughout the navigation path, Thus, the error contribution of the function under consideration can be deduced and compared to the system's error budget at a selected point. The same procedure is followed for the gravity gradients case. The results are listed below:

	Accelerometer	Gradiometer
Point	$\sigma_X^2=\sigma_Y^2=\sigma_Z^2:(m^2)$	$\sigma_X^2=\sigma_Y^2=\sigma_Z^2:(m^2)$
3	$0.10\ 10^{-5}$	$2.70\ 10^{-10}$
4	$0.50\ 10^{-5}$	$1.35\ 10^{-9}$
5	$0.22\ 10^{-4}$	$5.94\ 10^{-9}$
6	$0.94\ 10^{-4}$	$2.53\ 10^{-8}$
7	$0.40\ 10^{-3}$	$1.07\ 10^{-7}$
8	$0.17\ 10^{-2}$	$4.56\ 10^{-7}$
9	$0.72\ 10^{-2}$	$1.93\ 10^{-6}$
10	$0.31\ 10^{-1}$	$8.19\ 10^{-6}$

From the listed results it is evident that

1)the accelerometer error contribution into the system's error budget is about 7% of the total and

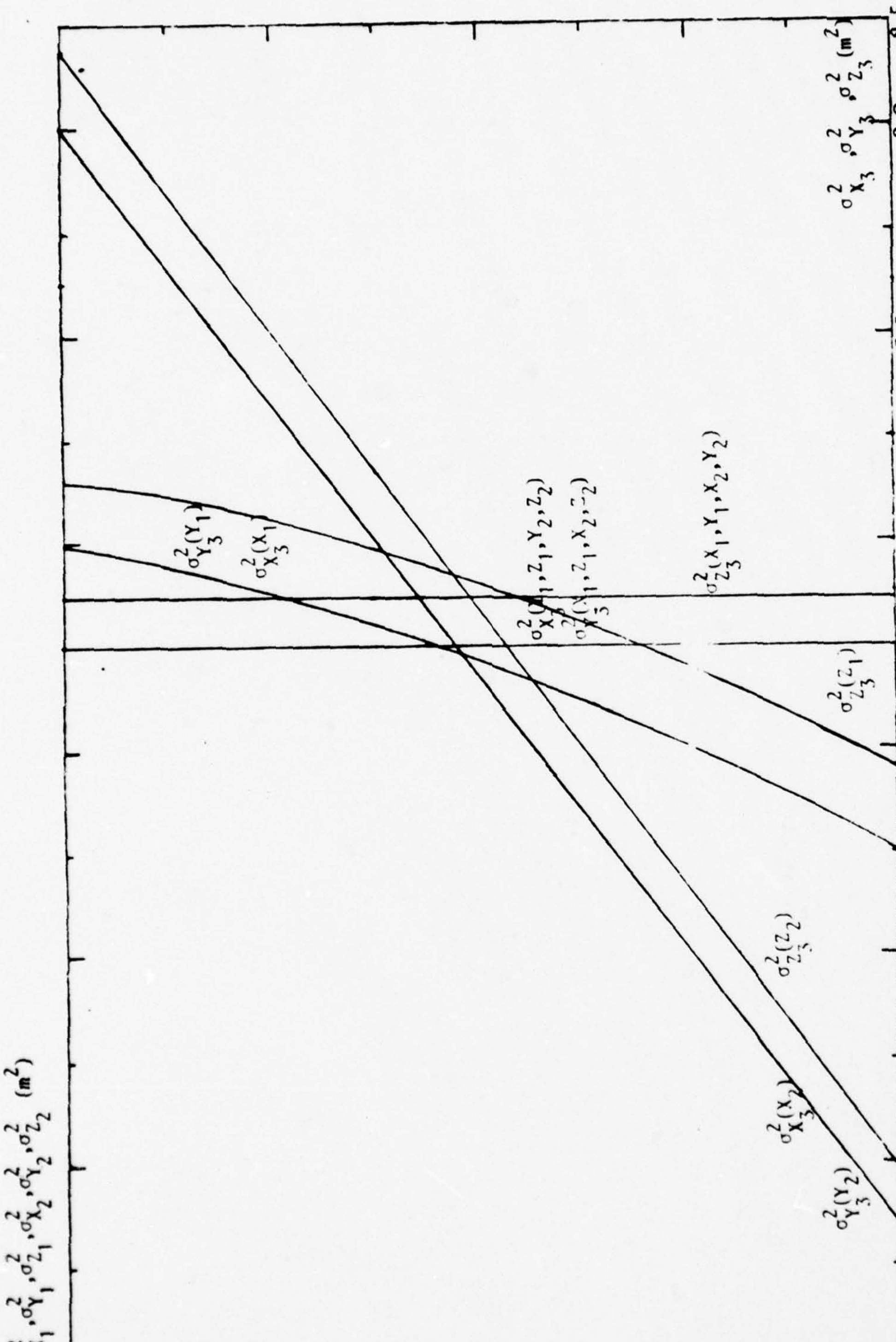
2)the gradiometers work perfectly well in such a navigation system. Their

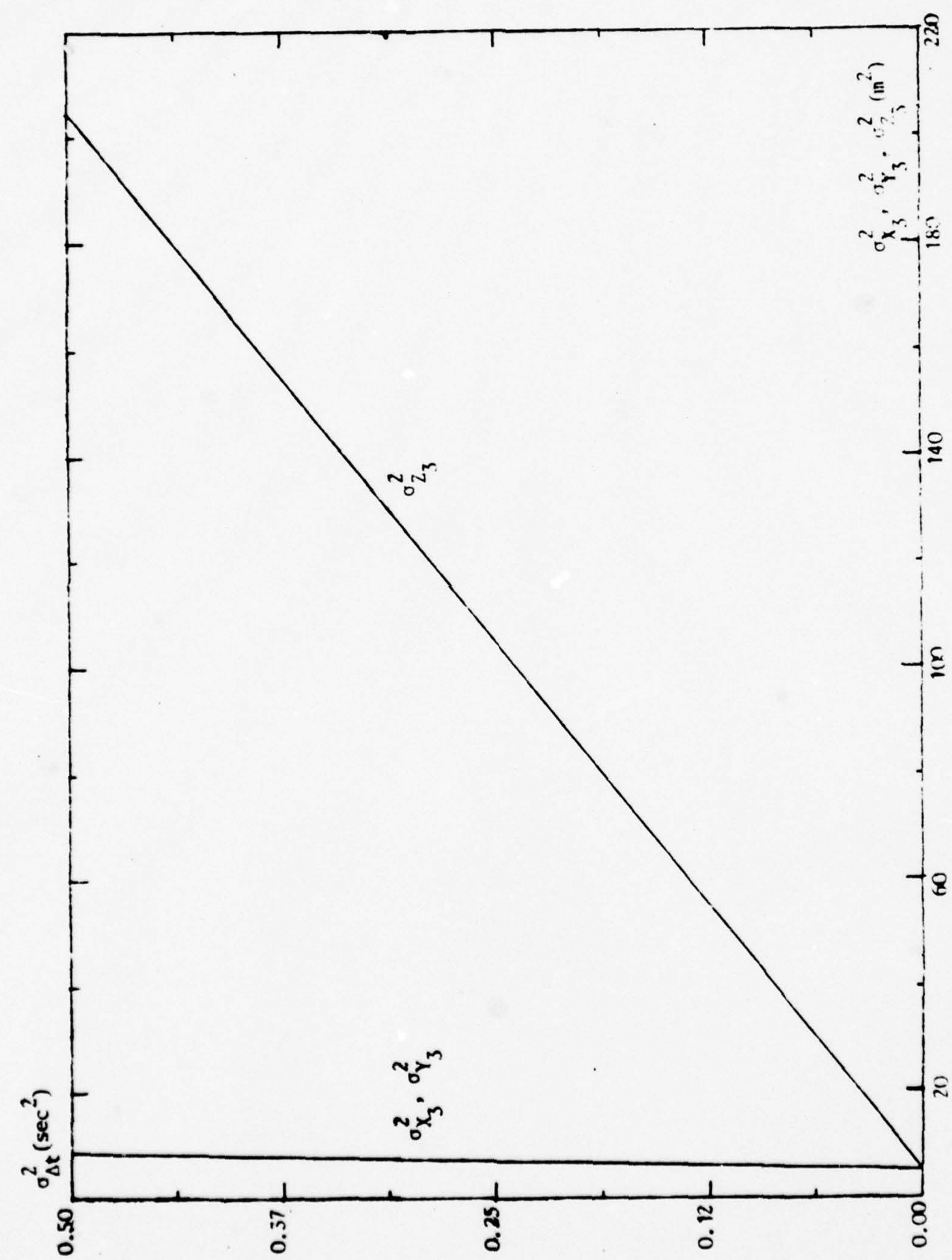
error contribution is really minimal.

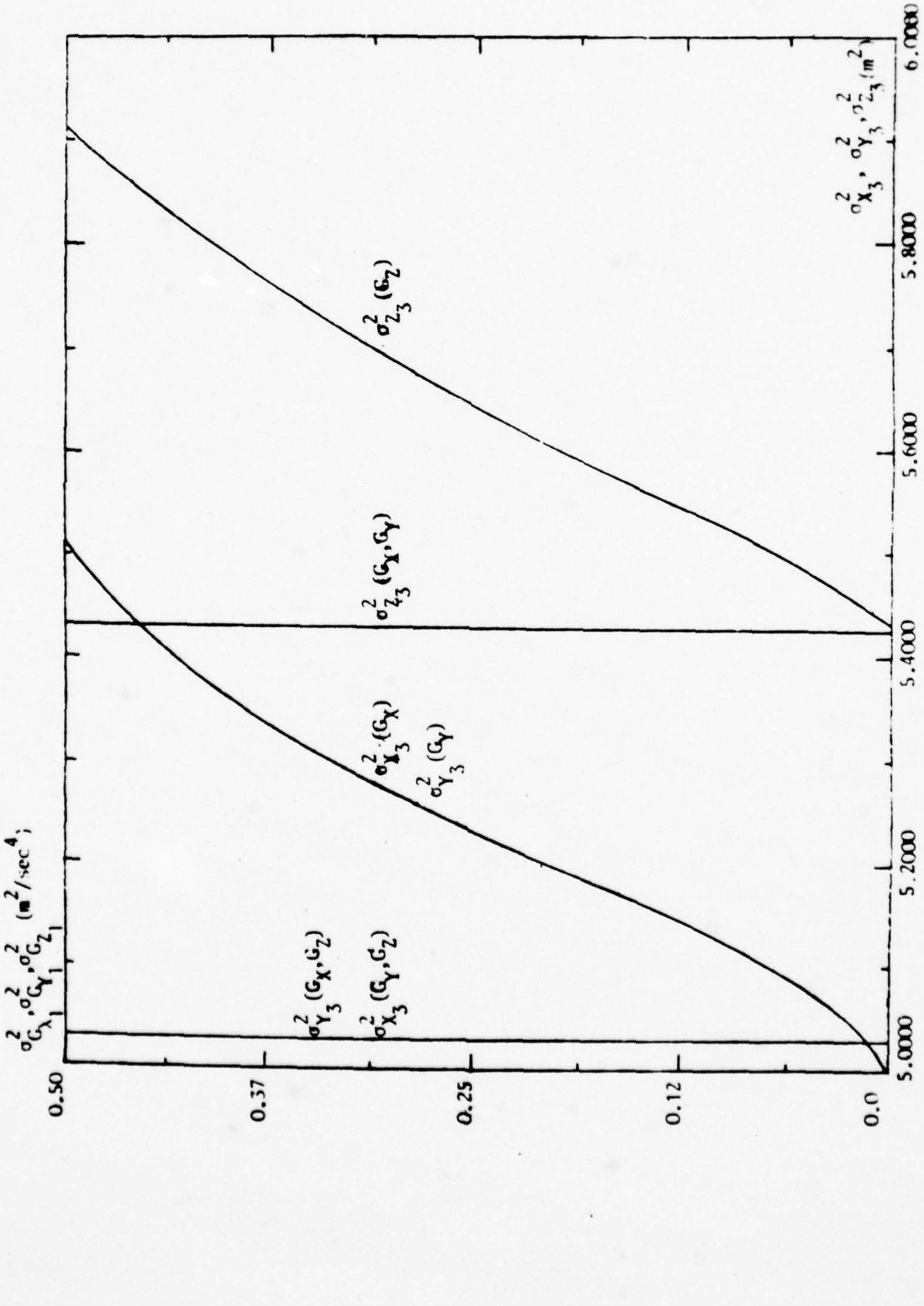
In the next pages we give some representative nomographs which illustrate the behaviour of the parameters of the system relevant to the obtained accuracy of the inertial coordinates.

In section 7 we will come again to discuss the same topic but this time detailed error models for the accelerometer and gravity gradients functions will be included to see if the system's results could be improved.

Explanatory note: *The following symbolism is used in the attached nomographs e.g. $\sigma^2_{X_3}(A_{X_{12}})$. It means that $\sigma^2_{X_3}$ undergoes the changes pictured by the respective line only when $A_{X_{12}}$ assumes a range of values.*


$\sigma^2_{X_1}, \sigma^2_{Y_1}, \sigma^2_{Z_1}, \sigma^2_{X_2}, \sigma^2_{Y_2}, \sigma^2_{Z_2}$ (m^2)
$\sigma^2_{X_3}, \sigma^2_{Y_3}, \sigma^2_{Z_3}$ (m^2)
2.0
1.5
1.0
0.5
0.0
0.0
1.5
3.0
4.5
6.0
7.5
9.0
9.5
$\sigma^2_{Y_3}(Y_1)$
$\sigma^2_{X_3}(X_1)$
$\sigma^2_{X_3}(X_1, Z_1, Y_2, Z_2)$
$\sigma^2_{Y_3}(Y_1, Z_1, X_2, Z_2)$
$\sigma^2_{Z_3}(X_1, Y_1, X_2, Y_2)$
$\sigma^2_{Z_3}(Z_1)$
$\sigma^2_{Z_3}(Z_2)$
$\sigma^2_{X_3}(X_2)$
$\sigma^2_{Y_3}(Y_2)$


0.50
0.37
0.25
0.12
0.00
$\sigma^2_{\Delta t}$ (sec^2)
$\sigma^2_{X_3}$, $\sigma^2_{Y_3}$
$\sigma^2_{Z_3}$
20
60
100
140
180
220
$\sigma^2_{X_3}$, $\sigma^2_{Y_3}$, $\sigma^2_{Z_3}$ (m^2)



$\sigma^2_{G_{X_1}}, \sigma^2_{G_{Y_1}}, \sigma^2_{G_{Z_1}}$ (m^2/sec^4)
0.50
0.37
0.25
0.12
0.0
5.0000
5.2000
5.4000
5.6000
5.8000
6.0000
$\sigma^2_{X_3}, \sigma^2_{Y_3}, \sigma^2_{Z_3}$ (m^2)
$\sigma^2_{Y_3}(G_X, G_Z)$
$\sigma^2_{X_3}(G_Y, G_Z)$
$\sigma^2_{X_3}(G_X)$
$\sigma^2_{Y_3}(G_Y)$
$\sigma^2_{Z_3}(G_X, G_Y)$
$\sigma^2_{Z_3}(G_Z)$

6. Inertial measurement unit quantization error analysis

6.1 The general equation

Quantization errors constitute a special error category in every instrumental package and must be clearly distinguished from round-off errors, commutation errors etc. Quantization errors arise from the fact that the inertial measurement units measure continuous physical quantities such as, say, the apparent acceleration vector components. Assuming that at each observation point at which the inertial instruments are read a quantization error exists, then its additive net effect will burden the desired output of the navigation system with an additional error. In our analysis, the system computes geocentric coordinates making use of accelerometer and gradiometer measurements. Expressing position with respect to the measurables and attaching to them different quantization errors, then the position error could be computed treating the quantization errors as stochastic quantities. Regarding the up-to-date accuracy of the inertial navigation systems, it will be seen that the quantization-induced position error is not of great importance(due to its small magnitude). But, there is one case in which that error could be of great interest. As we have already seen, all inertial navigation systems without exception must be initially aligned to a desirable coordinate system prior to their mission. In some cases, the initial alignment procedure can consume an appreciable long time(especially in commercial flights) and during that time the inertial instrumentation is continuously operating and, of course, gathering quantization errors. These errors could cause an initial misalignment error which is carried through the entire mission.

Consequently, it is our belief that quantization errors must be always considered to preestimate how badly they burden the system's output. In what follows, we present the first unique gradiometer-aided inertial navigation systems quantization error analysis using the fundamental equation of inertial navigation used in the preceding simulation.

As we have already seen, the formula which estimates the inertial coordinates is written in the form:

(6.1.1) $\ddot{P}_n = A_n + G_n$

where A and G the non-gravitational and gravitational acceleration respectively, P indicates position(in eq.(5.2.1) R is used instead of P) and n represents the n^{th}-observation point. Stirling's formula approximates the inertial acceleration as:

(6.1.2) $\ddot{P}_n = \Delta t^{-2}(P_{n+1} - 2P_n + P_{n-1})$

Eq.(6.1.2) solved for the n^{th}-inertial coordinate gives

(6.1.3) $P_n = 2P_{n-1} - P_{n-2} + \Delta t^2 A_n + \Delta t^2 G_n$

Our policy is now to express the above equation as a function of the first two inertial coordinates which are then to be considered as errorless quantities. In order to find the general formula which gives the coordinates of the n^{th}-point, we write

(6.1.4)
$$P_3 = 2P_2 - P_1 + \Delta t^2 A_2 + \Delta t^2 G_2$$
$$P_4 = 3P_2 - 2P_1 + \Delta t^2(2A_2 + A_3) + \Delta t^2(2G_2 + G_3)$$
.....

Consequently, we get

(6.1.5)
$$P_n = (n-1)P_2 - (n-2)P_1 + \Delta t^2((n-2)A_2 + (n-3)A_3 + \cdots + (n-(n-1))A_{n-1}) + \Delta t^2((n-2)G_2 + (n-3)G_3 + \cdots + (n-(n-1))G_{n-1})$$

and in summation form

(6.1.6) $P_n = (n-1)P_2 - (n-2)P_1 + \sum_{m=2}^{n-1}(n-m)\{A_m + G_m\}\Delta t^2$

Now, employing the free-coordinate approximation for the gravity gradients we write

(6.1.7) $G_m = G_{m-1} + \mathrm{grad}G_{m-1}(P_m - P_{m-1})$

Expressing the above equation in terms of the gravity components of the first point plus other gravity gradients, we get

$$(6.1.8)\qquad G_m=G_1+\sum_{k=2}^{m}\mathrm{grad}G_{k-1}(P_k-P_{k-1})$$

Consequently, eq.(6.1.6) is now written

$$(6.1.9)\qquad P_n=(n-1)P_2-(n-2)P_1+\sum_{m=2}^{n-1}(n-m)\{A_m+G_1+\sum_{k=2}^{m}\mathrm{grad}G_{k-1}(P_k-P_{k-1})\}\,\Delta t^2$$

Assuming now that each time the accelerometers and gradiometers are read a quantization error is committed, we express

$$(6.1.10)\qquad P_n+\delta P_n=(n-1)P_2-(n-2)P_1+\sum_{m=2}^{n-1}(n-m)(A_m+q_m^a)\Delta t^2+G_1+\sum_{m=2}^{n-1}(n-m)\sum_{k=2}^{m}(\mathrm{grad}G_{k-1}(P_k-P_{k-1})+q_{k-1}^g)\Delta t^2$$

where δP_n :quantization-induced position error

q_m^a :accelerometer quantization error

q_{k-1}^g:gradiometer quantization error

Taking into consideration only the impact of the quantization errors, then the position error assumes the form

$$(6.1.11)\qquad \delta P_n=\Delta t^2\sum_{m=2}^{n-1}(n-m)q_m^a+\Delta t^2\sum_{m=2}^{n-1}(n-m)\sum_{k=2}^{m}q_{k-1}^g$$

We shall derive now the general relation for the variance-covariance of the position error δP_n . Generally, we can write

$$(6.1.12)\qquad \mathrm{cov}(\delta P_n)=E\{\delta P_n-E(\delta P_n)\}\{\delta P_m-E(\delta P_m)\}$$

which yields

$$(6.1.13)\qquad \begin{aligned}\mathrm{cov}(\delta P_n)=E\{&\sum_{m=2}^{n-1}(n-m)\Delta t^2(q_m^a-E\{q_m^a\})+\sum_{m=2}^{n-1}(n-m)\sum_{k=2}^{m}\Delta t^2(q_{k-1}^g-E\{q_{k-1}^g\})\}\cdot E\{\sum_{r=2}^{s-1}(s-r)\Delta t^2(q_r^a-\\ &-E\{q_r^a\})+\sum_{r=2}^{s-1}(s-r)\sum_{p=2}^{r}\Delta t^2(q_{p-1}^g-E\{q_{p-1}^g\})\}\\ =&\sum_{m=2}^{n-1}\sum_{r=2}^{s-1}(n-m)(s-r)\Delta t^4D(q_m^a,q_r^a)+\sum_{m=2}^{n-1}\sum_{r=2}^{s-1}\sum_{k=2}^{m}\sum_{p=2}^{r}(n-m)(s-r)\Delta t^4D(q_{k-1}^g,q_{p-1}^g)+\\ &+2\sum_{m=2}^{n-1}\sum_{r=2}^{s-1}\sum_{p=2}^{r}(n-m)(s-r)\Delta t^4D(q_m^a,q_{p-1}^g)\end{aligned}$$

where D( ) represents the dispersion of the enclosed quantities. But, as we have already discussed, the gradiometer measurement unit is completely separated from the accelerometers and gyros units' platform. Consequently, it is reasonable to declare that the gradiometer and accelerometer quantization errors are not correlated and therefore, the last term of eq.(6.1.13) is dropped. For the dispersion of the quantization errors an exponentially decreasing correlation function is assumed of type(see Denhard, 1977)

$$D(a_i,a_j)=\sigma^2\, e^{-\frac{|t_i-t_j|}{\tau}} \tag{6.1.14}$$

where σ^2 the variance of each measurable, t_i-t_j the sampling interval and τ the correlation coefficient. Therefore, taking into account that we have different correlation coefficient between accelerometer unit(τ_a) and gradiometer unit(τ_g), then the variance-covariance expressions are given:

$$var(\delta P_n)=\Delta t^4\sigma_a^2 \sum_{m=2}^{n-1}\sum_{n=2}^{s-1}(n-m)(s-n)e^{-\frac{|t_m-t_n|}{\tau_a}}+\Delta t^4\sigma_g^2\sum_{m=2}^{n-1}\sum_{n=2}^{s-1}(n-m)(s-n)\cdot \sum_{k=2}^{m}\sum_{p=2}^{n} e^{-\frac{|t_{k-1}-t_{p-1}|}{\tau_g}}$$

(6.1.15)

$$cov(\delta P_n,\delta P_r)=\Delta t^4\sigma_a^2 \sum_{m=2}^{n-1}\sum_{r=2}^{s-1}(n-m)(s-r)e^{-\frac{|t_m-t_r|}{\tau_a}}+\Delta t^4\sigma_a^2\sum_{m=2}^{n-1}\sum_{r=2}^{s-1}(n-m)(s-r)\cdot \sum_{k=2}^{m}\sum_{r=2}^{r} e^{-\frac{|t_{k-1}-t_{p-1}|}{\tau_g}}$$

where σ_a^2, σ_g^2 the accelerometer and gradiometer variances respectively. Eqs(6.1.15) are our simulation equations to find the order of magnitude of the quantization-induced position error.

6.2 Results

Two computer programmes have been written and compiled in the Siemens Computer, the first to examine the variance case and the second the co-variance one. These two programmes are listed in Appendix A. The ground values for the five parameters of the above equations have been chosen to be:

$$\Delta t = 0.1 \text{ sec.}$$

$$\sigma_a^2 = 0.01 \text{ m}^2/\text{sec}^4$$

$$\sigma_g^2 = 1. \text{ E.}$$

$$\tau_a = 0.001 \text{ sec.}$$

$$\tau_g = 0.001 \text{ sec.}$$

The same statistical technique used in simulation I, is followed again for the quantization error simulation studies. Four of the five parameters have been kept constant and at the same time the fifth one assumed a range of possible values. The computer results are given below:

Time	σ_g^2=1E	20E	50E	100E	500E	1000E
	Error(m^2)					
1sec	$0.6400\ 10^{-6}$	"	"	"	"	"
2	$0.3240\ 10^{-5}$	"	"	"	"	"
3	$0.7840\ 10^{-5}$	"	"	"	"	"
5	$0.2304\ 10^{-4}$	"	"	"	"	"
10	$0.9604\ 10^{-4}$	"	"	"	"	"
20	$0.3920\ 10^{-3}$	"	"	"	"	"
30	$0.8880\ 10^{-3}$	"	"	"	"	"
40	$0.1584\ 10^{-2}$	"	"	"	"	"
50	$0.2480\ 10^{-2}$	"	"	"	"	"
60	$0.3576\ 10^{-2}$	"	"	"	"	"

Time	σ_a^2=0.001m^2/sec^4	0.005m^2/sec^4	0.01m^2/sec^4	0.02m^2/sec^4	0.05m^2/sec^4	0.1m^2/sec^4
	Error(m^2)					
1sec	$0.6400\ 10^{-8}$	$0.1600\ 10^{-6}$	$0.6400\ 10^{-6}$	$0.2560\ 10^{-5}$	$0.1600\ 10^{-4}$	$0.6400\ 10^{-4}$
2	$0.3240\ 10^{-7}$	$0.8100\ 10^{-6}$	$0.3240\ 10^{-5}$	$0.1296\ 10^{-4}$	$0.8100\ 10^{-4}$	$0.3240\ 10^{-3}$
3	$0.7840\ 10^{-7}$	$0.1960\ 10^{-5}$	$0.7840\ 10^{-5}$	$0.3136\ 10^{-4}$	$0.1960\ 10^{-3}$	$0.7840\ 10^{-3}$
5	$0.2304\ 10^{-6}$	$0.5760\ 10^{-5}$	$0.2304\ 10^{-4}$	$0.9216\ 10^{-4}$	$0.5760\ 10^{-3}$	$0.2304\ 10^{-2}$
10	$0.9604\ 10^{-6}$	$0.2401\ 10^{-4}$	$0.9604\ 10^{-4}$	$0.3842\ 10^{-3}$	$0.2401\ 10^{-2}$	$0.9604\ 10^{-2}$
20	$0.3920\ 10^{-5}$	$0.9801\ 10^{-4}$	$0.3920\ 10^{-3}$	$0.1568\ 10^{-2}$	$0.9801\ 10^{-2}$	$0.3920\ 10^{-1}$
30	$0.8880\ 10^{-5}$	$0.2220\ 10^{-3}$	$0.8880\ 10^{-3}$	$0.3552\ 10^{-2}$	$0.2220\ 10^{-1}$	$0.8880\ 10^{-1}$
40	$0.1584\ 10^{-4}$	$0.3960\ 10^{-3}$	$0.1584\ 10^{-2}$	$0.6336\ 10^{-2}$	$0.3960\ 10^{-1}$	0.1584
50	$0.2480\ 10^{-4}$	$0.6200\ 10^{-3}$	$0.2480\ 10^{-2}$	$0.9920\ 10^{-2}$	$0.6200\ 10^{-1}$	0.2480
60	$0.3576\ 10^{-4}$	$0.8940\ 10^{-3}$	$0.3576\ 10^{-2}$	$0.1430\ 10^{-1}$	$0.8940\ 10^{-1}$	0.3576

$0.10\ 10^{-9}$ $0.10\ 10^{-5}$	$0.20\ 10^{-9}$ $0.20\ 10^{-5}$	$0.30\ 10^{-9}$ $0.30\ 10^{-5}$	$0.40\ 10^{-9}$ $0.40\ 10^{-5}$	$0.50\ 10^{-9}$ $0.50\ 10^{-5}$	$0.60\ 10^{-9}$ $0.60\ 10^{-5}$	$0.70\ 10^{-9}$ $0.70\ 10^{-5}$	$0.80\ 10^{-9}$ $0.80\ 10^{-5}$
	$0.50\ 10^{-9}$ $0.50\ 10^{-5}$	$0.80\ 10^{-9}$ $0.80\ 10^{-5}$	$0.11\ 10^{-8}$ $0.11\ 10^{-4}$	$0.14\ 10^{-8}$ $0.14\ 10^{-4}$	$0.17\ 10^{-8}$ $0.17\ 10^{-4}$	$0.20\ 10^{-8}$ $0.20\ 10^{-4}$	$0.23\ 10^{-8}$ $0.23\ 10^{-4}$
		$0.14\ 10^{-8}$ $0.14\ 10^{-4}$	$0.20\ 10^{-8}$ $0.20\ 10^{-4}$	$0.26\ 10^{-8}$ $0.26\ 10^{-4}$	$0.32\ 10^{-8}$ $0.32\ 10^{-4}$	$0.38\ 10^{-8}$ $0.38\ 10^{-4}$	$0.44\ 10^{-8}$ $0.44\ 10^{-4}$
			$0.30\ 10^{-8}$ $0.30\ 10^{-4}$	$0.40\ 10^{-8}$ $0.40\ 10^{-4}$	$0.50\ 10^{-8}$ $0.50\ 10^{-4}$	$0.60\ 10^{-8}$ $0.60\ 10^{-4}$	$0.70\ 10^{-8}$ $0.70\ 10^{-4}$
				$0.55\ 10^{-8}$ $0.55\ 10^{-4}$	$0.70\ 10^{-8}$ $0.70\ 10^{-4}$	$0.85\ 10^{-8}$ $0.85\ 10^{-4}$	$0.10\ 10^{-7}$ $0.10\ 10^{-3}$
					$0.91\ 10^{-8}$ $0.91\ 10^{-4}$	$0.11\ 10^{-7}$ $0.11\ 10^{-3}$	$0.13\ 10^{-7}$ $0.13\ 10^{-3}$
						$0.14\ 10^{-7}$ $0.14\ 10^{-3}$	$0.17\ 10^{-7}$ $0.17\ 10^{-3}$
							$0.20\ 10^{-7}$ $0.20\ 10^{-3}$

Acceleration-induced quantization error. The first rows represent σ_a=0.001 and the second ones $\sigma_a=0.1 m^2/sec^4$ (Covariance case)

Time				Δt=a)0.01sec Error(m^2)	b)0.1sec	c)0.5sec	d)1sec
a)0.1sec	b)1sec	c)5sec	d)10sec	$0.6400\ 10^{-10}$	$0.6400\ 10^{-6}$	$0.4000\ 10^{-3}$	$0.6400\ 10^{-2}$
0.2	2	10	20	$0.3240\ 10^{-9}$	$0.3240\ 10^{-5}$	$0.2025\ 10^{-2}$	$0.3240\ 10^{-1}$
0.3	3	15	30	$0.7840\ 10^{-9}$	$0.7840\ 10^{-5}$	$0.4900\ 10^{-2}$	$0.7840\ 10^{-1}$
0.5	5	25	50	$0.2304\ 10^{-8}$	$0.2304\ 10^{-4}$	$0.1440\ 10^{-1}$	0.2304
1	10	50	100	$0.9604\ 10^{-8}$	$0.9604\ 10^{-4}$	$0.6003\ 10^{-1}$	0.9604
2	20	100	200	$0.3920\ 10^{-7}$	$0.3920\ 10^{-3}$	0.2450	3.9200
3	30	150	300	$0.8880\ 10^{-7}$	$0.8880\ 10^{-3}$	0.5550	8.8800
4	40	200	400	$0.1584\ 10^{-6}$	$0.1584\ 10^{-2}$	0.9900	15.840
5	50	250	500	$0.2480\ 10^{-6}$	$0.2480\ 10^{-2}$	1.5500	24.800
6	60	300	600	$0.3576\ 10^{-6}$	$0.3576\ 10^{-2}$.	22.3500	35.760

$0.10\ 10^{-7}$	$0.20\ 10^{-7}$	$0.30\ 10^{-7}$	$0.40\ 10^{-7}$	$0.50\ 10^{-7}$	$0.60\ 10^{-7}$	$0.70\ 10^{-7}$	$0.80\ 10^{-7}$
	$0.50\ 10^{-7}$	$0.80\ 10^{-7}$	$0.11\ 10^{-6}$	$0.14\ 10^{-6}$	$0.17\ 10^{-7}$	$0.20\ 10^{-7}$	$0.23\ 10^{-7}$
		$0.14\ 10^{-6}$	$0.20\ 10^{-6}$	$0.26\ 10^{-6}$	$0.32\ 10^{-6}$	$0.38\ 10^{-6}$	$0.44\ 10^{-6}$
			$0.30\ 10^{-6}$	$0.40\ 10^{-6}$	$0.50\ 10^{-6}$	$0.60\ 10^{-6}$	$0.70\ 10^{-6}$
				$0.55\ 10^{-6}$	$0.7o\ 10^{-6}$	$0.85\ 10^{-6}$	$0.10\ 10^{-5}$
					$0.91\ 10^{-6}$	$0.11\ 10^{-5}$	$0.13\ 10^{-5}$
						$0.14\ 10^{-5}$	$0.17\ 10^{-5}$
							$0.20\ 10^{-5}$

Gravity gradients-induced quantization error (the same output matrix from 1E -1000E). The same results for τ_g. (Covariance case)

$0.10\ 10^{-7}$ $0.10\ 10^{-7}$	$0.20\ 10^{-7}$ $0.29\ 10^{-7}$	$0.30\ 10^{-7}$ $0.56\ 10^{-7}$	$0.40\ 10^{-7}$ $0.91\ 10^{-7}$	$0.50\ 10^{-7}$ $0.13\ 10^{-6}$	$0.60\ 10^{-7}$ $0.18\ 10^{-6}$	$0.70\ 10^{-7}$ $0.23\ 10^{-6}$	$0.80\ 10^{-7}$ $0.29\ 10^{-6}$
	$0.50\ 10^{-7}$ $0.86\ 10^{-7}$	$0.80\ 10^{-7}$ $0.17\ 10^{-6}$	$0.11\ 10^{-6}$ $0.27\ 10^{-6}$	$0.14\ 10^{-6}$ $0.40\ 10^{-6}$	$0.17\ 10^{-6}$ $0.55\ 10^{-6}$	$0.20\ 10^{-6}$ $0.71\ 10^{-6}$	$0.23\ 10^{-6}$ $0.89\ 10^{-6}$
		$0.14\ 10^{-6}$ $0.33\ 10^{-6}$	$0.20\ 10^{-6}$ $0.55\ 10^{-6}$	$0.26\ 10^{-6}$ $0.80\ 10^{-6}$	$0.32\ 10^{-6}$ $0.11\ 10^{-5}$	$0.38\ 10^{-6}$ $0.14\ 10^{-5}$	$0.44\ 10^{-6}$ $0.18\ 10^{-5}$
			$0.30\ 10^{-6}$ $0.90\ 10^{-6}$	$0.40\ 10^{-6}$ $0.13\ 10^{-5}$	$0.50\ 10^{-6}$ $0.18\ 10^{-5}$	$0.60\ 10^{-6}$ $0.24\ 10^{-5}$	$0.70\ 10^{-6}$ $0.30\ 10^{-5}$
				$0.55\ 10^{-6}$ $0.20\ 10^{-5}$	$0.70\ 10^{-6}$ $0.27\ 10^{-5}$	$0.85\ 10^{-6}$ $0.36\ 10^{-5}$	$0.10\ 10^{-5}$ $0.45\ 10^{-5}$
					$0.91\ 10^{-6}$ $0.38\ 10^{-5}$	$0.11\ 10^{-5}$ $0.50\ 10^{-5}$	$0.13\ 10^{-5}$ $0.65\ 10^{-5}$
						$0.14\ 10^{-5}$ $0.65\ 10^{-5}$	$0.17\ 10^{-5}$ $0.83\ 10^{-5}$
							$0.20\ 10^{-5}$ $0.11\ 10^{-4}$

Correlation coefficient-induced quantization error. The first rows represent τ_a=0.002 and the second ones τ_a=1 sec. (Covariance case)

6.3 <u>Discussion</u>

For the given results the following comments summarize their meaning:
a) comparing the errors commited by the gradiometer-aided navigation system with those coming from the quantization studies, it can be clearly seen that the latter errors constitute a very small quantity. The results can be generally considered as satisfactory in view of the fact that the approximation formula used for the inertial acceleration components approximates grossly the reality using only three points. If more terms in the Stirling's formula are taken into account, then the quantization error studies' results would be effectively reduced.
b) the sampling interval Δt is the worst contributing error factor in the quantization error. Decreasing the operation of the system to the order of 10(ten), then the quantization errors increase up to the 10^3.
c) for the rest four parameters of the system only the accelerometer variance causes changes in the quantization error budget. For example,

decreasing the variance of the measured acceleration 10 times, then the quantization errors increase up to the 10^2.

d)when the parameters σ_g, τ_a, τ_g undergo their range of changes, the quantization errors remain unaffected. For that reason we have listed only the σ_g contribution with the understanding that the rest two parameters give identical results

e)comments b) and c) are also valid for the covariance case which is included herein for instructive purposes.

In the next pages we give some representative nomographs to picture briefly the quantization error studies results.

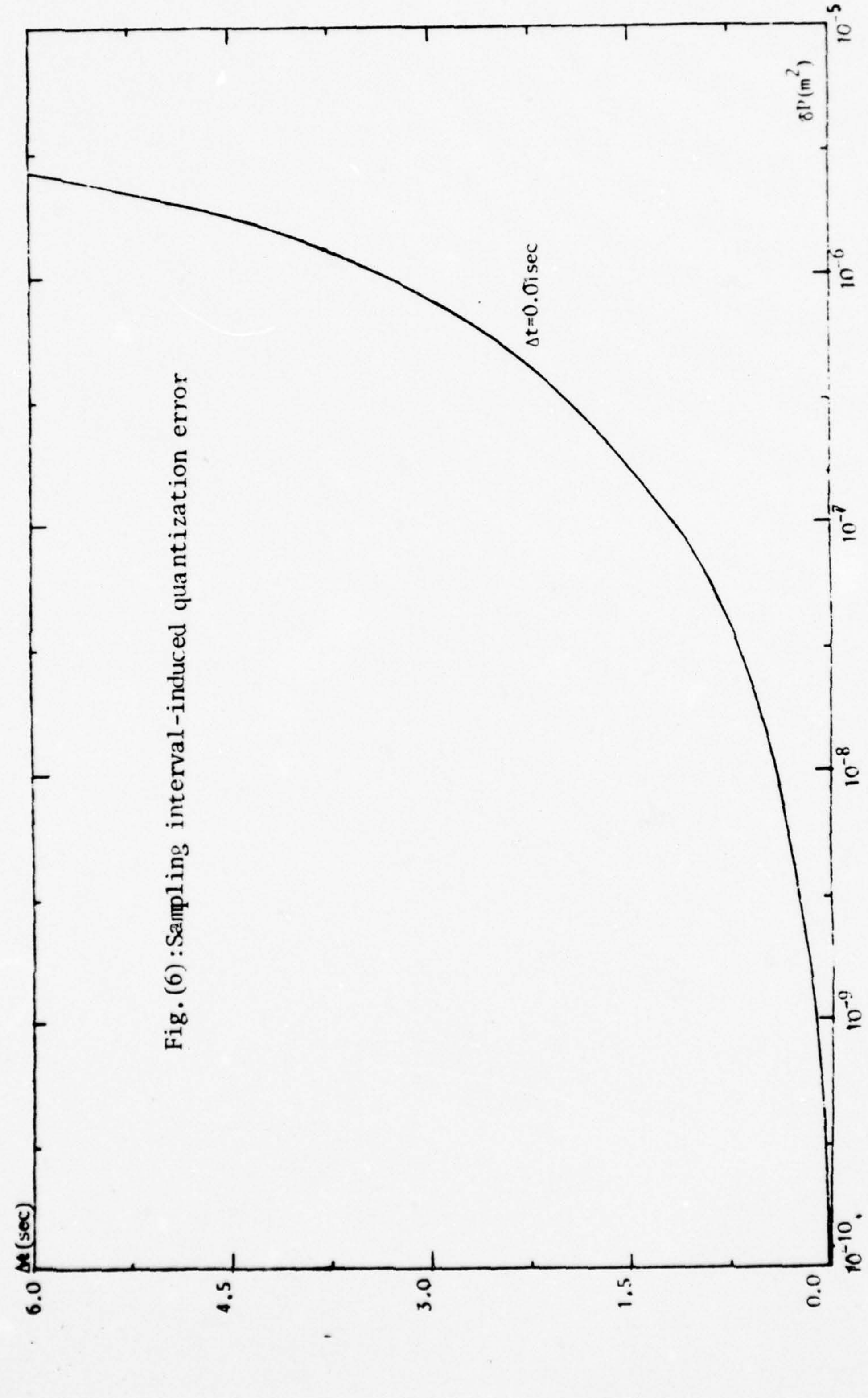


Fig. (6):Sampling interval-induced quantization error

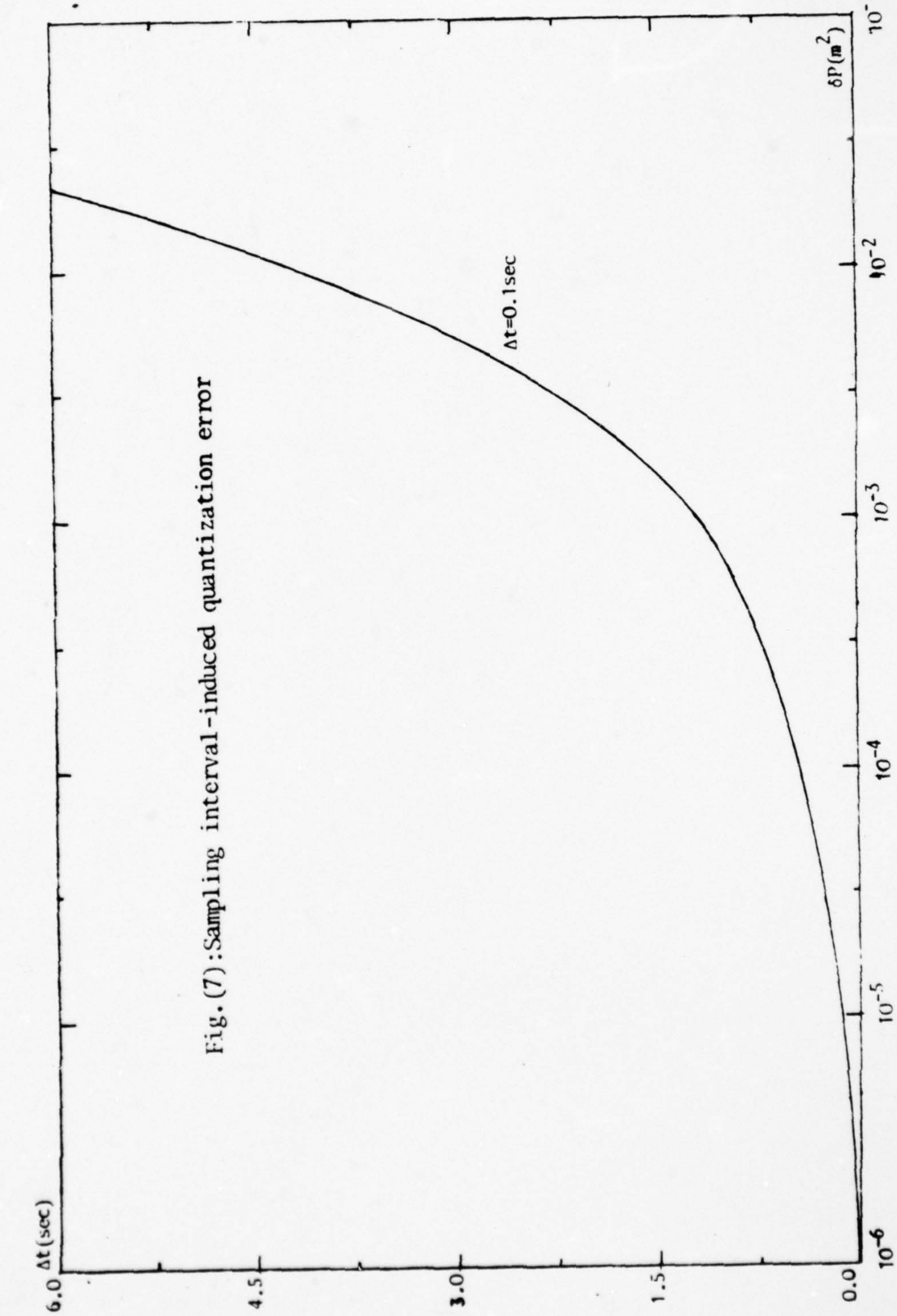


Fig. (7):Sampling interval-induced quantization error

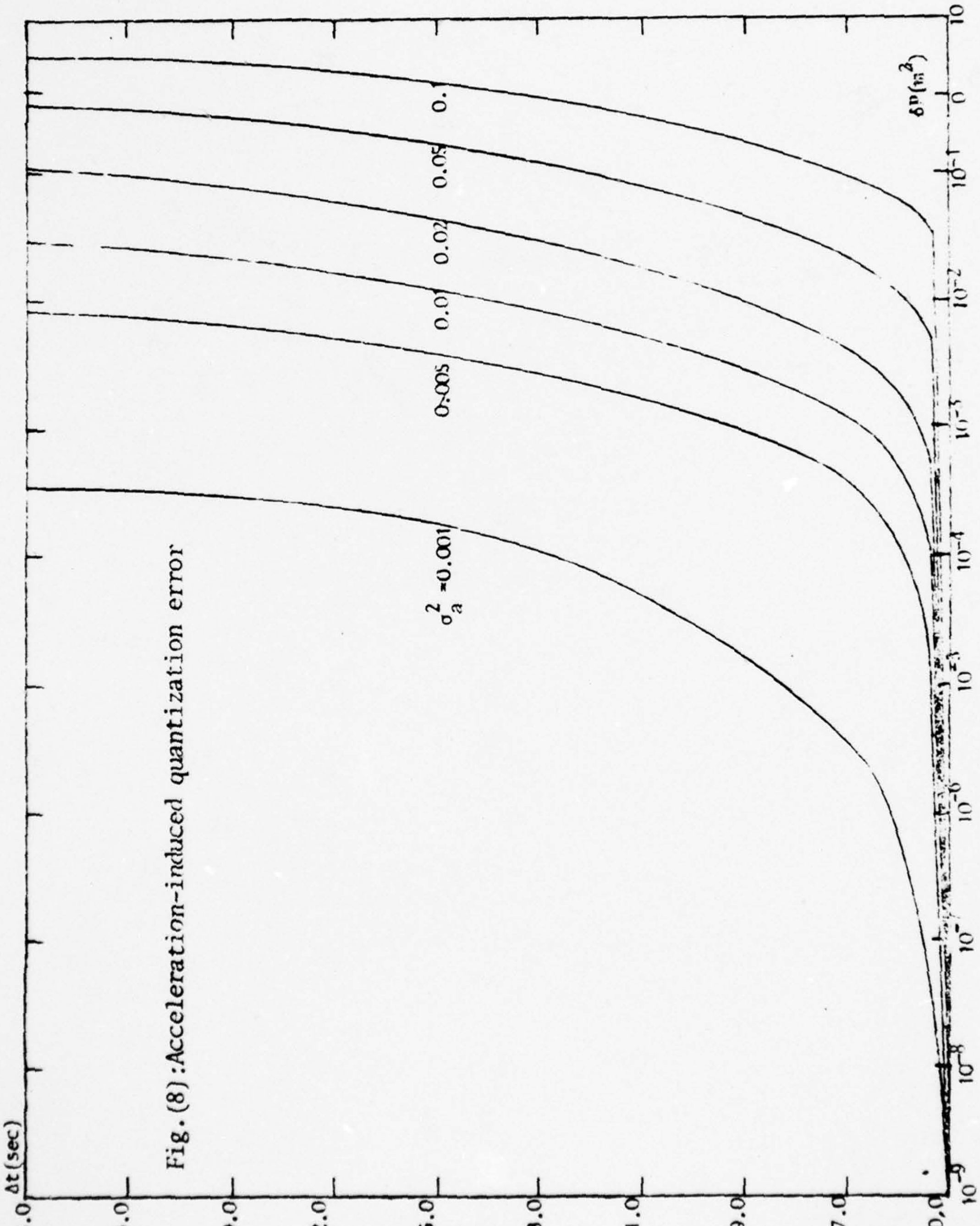


Fig. (8):Acceleration-induced quantization error

7. Simulation II: The General Error Model

7.1 Accelerometer Error Studies

It is well-known that the accelerometer frame is materialized by the three input axes of the on-board accelerometers. Since it is instrumentally impossible to direct three axes so as to construct an orthogonal frame, the accelerometer frame is finally a non-orthogonal or quasi-orthogonal frame. Consequently, the fact of measuring the apparent acceleration components along a non-orthogonal frame should be seriously taken into account. Having corrected the sensed acceleration for accelerometer non-orthogonality, then it refers to the actual platform frame. A transformation which takes the acceleration signal from the actual platform frame to the ideal one, is the next step to be accomplished. All gyro misfunctions are included in the aforementioned transformation. As soon as the apparent acceleration refers to the ideal platform frame, which in our case coincides with the navigation frame, then its components can enter the general equation of inertial navigation.

Taking into consideration what is discussed above, the apparent acceleration signal transformation could be illustrated by the general representation

$$A^N = C^{P^i}_{P^a} C^{P^a}_{a} A^a \tag{7.1.1}$$

where A^N , A^a represent the acceleration signal coordinatized in the navigation and accelerometer frames respectively and C represents the direction cosine matrix(from where the notation comes) which transforms the frame indicated by the subscript to that indicated by the superscript. P^a and P^i denote the actual and ideal platform frames respectively.

Since we are using very often transformations of type affine and similarity ones, it would be helpful to define them from the beginning
a)the group of the affine transformations can be represented by a rotation matrix H plus a transformation vector t, that is

$$T(u)=Hu+t \tag{7.1.2}$$

(see Grafarend and Schaffrin, 1976). The affine transformation preserves Euclidean parallelism, straight lines are transformed into straight lines and planes into planes.

b)when the two relations

$$(7.1.3)\qquad \begin{aligned} H &= \lambda R \\ R^{-1}R &= I \end{aligned}$$

hold, then the affine transformation group is called similarity transformation group and under that ratios of distances and angles are preserved.

In our analysis, skew-symmetric matrices, denoted by R_a^b, are very often used to transform two misaligned orthogonal coordinate frames into each other in case the misalignment angles are considered small. It helps in the understanding of what follows to note that skew-symmetric matrices are always transformed under the similarity group.

Now, we shall try to determine the two transformation matrices involved in eq.(7.1.1) taking into account the error sources which cause them to depart from the identity matrix.

1. $C_a^{P^a}$ - transformation

The transformation between the quasi-orthogonal accelerometer frame a and the orthogonal actual platform frame P^a is a "small angle" transformation parametarized by the small angle rotations connecting the two frames. This transformation is treated in many textbooks in detail and it will not be further considered herein(for a discussion see Britting, 1971, p.39). Taking into account the angles definition depicted in Fig. (9), we write

$$(7.1.4)\qquad A^{P^a} = C_a^{P^a}\, A^a = \begin{bmatrix} 1 & -\varepsilon_{XZ} & \varepsilon_{XY} \\ \varepsilon_{YZ} & 1 & -\varepsilon_{YX} \\ -\varepsilon_{ZY} & \varepsilon_{ZX} & 1 \end{bmatrix} A^a$$

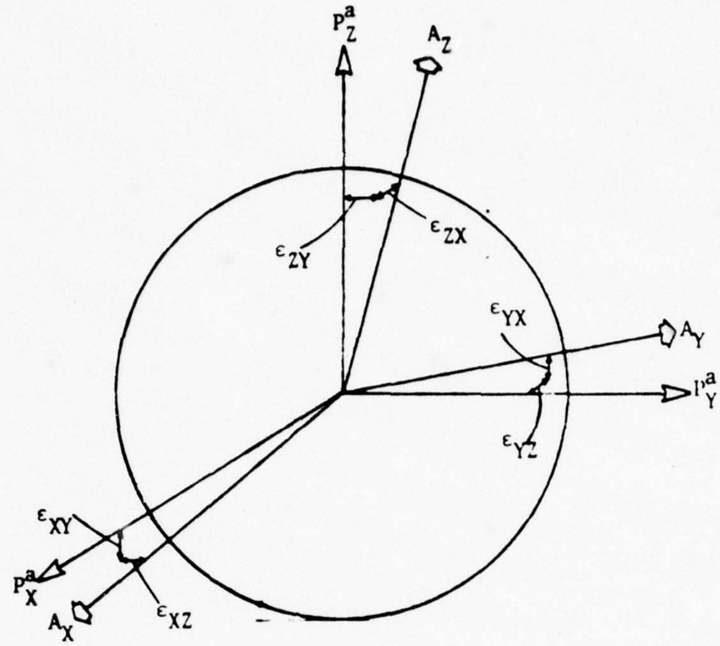


Fig. (9): Actual platform, accelerometer frames geometry.

The only problem on which we like to draw attention concerning the above transformation is that the angles inside the $C_a^{p^a}$ matrix are very small as well as the off diagonal terms are not equal, e.g. $\varepsilon_{XZ} \neq \varepsilon_{YZ}$. The justification for that comes from the non-orthogonality of the accelerometer frame. Consequently, a small rotation about, say, the a_X-axis first and then about a_Y and a_Z will not be sufficient to bring the accelerometer frame in coincidence with the actual platform. We note, finally, that the six angles depicted in eq.(7.1.4) can be measured by well-known alignment techniques.

2. $C_{p^a}^{p^i}$ - transformation

As we said before, the $C_{p^a}^{p^i}$ -transformation is by far the most critical operational procedure in the whole navigation systems analysis. Deeply thinking, what is written in the literature known to the writer could be considered as a mess as far as this transformation is concerned. Consequently, we feel that it is our turn to put things into an order by making from the very beginning the following statements:

a) When the moving vehicle is at the starting point we have to decide which will be the computation-navigation-ideal platform frame. In our analysis, the decision was taken in favour of the Greenwich orthogonal frame and it was inscribed on the moving platform everafter.

b) In order to have at any time instant the ideal platform frame parallel to the navigation frame, the former is commanded to the earth's rotation.

c) The following statement has no impact on the mathematical analysis of the problem under consideration, but it has to be made in order to give rigor and clarity to the general concept: an inertial coordinate frame is somehow and somewhere inscribed on the moving platform and we refer to it when we postulate that the platform rotates. The first idea to be accomplished is to materialize such an inertial frame by a set of three single-degree-of-freedom comoving but inertially stabilized gyros. Otherwise, who can insist on saying that the moving platform is inertially rotating?

d) The command for platform rotation equals to the earth's rotation is injected to the gyros which drive then the platform accordingly. But since the gyros, like all other instrumental units, are burdened with a variety of serious errors e.g. gyro drift, they have to be plugged into the $C_{p^a}^{p^i}$ - transformation.

e) At the starting point, the platform frame is set to be parallel to the navigation frame. Of course, this is by no means true and thus an initial misalignment is everafter present.

Let us now proceed in determining the discussed transformation. As it is shown in Fig. (10), at the starting point (t=0) the actual platform frame P^a has a small initial misalignment with respect to the ideal platform frame P^i due to the errors in the alignment procedure. These two frames seen at another time instant t have already changed their respective attitude due only to the inabilities of the gyros. The actual platform frame, besides its initial misalignment, has already got another small angle distortion, time dependent one, denoted by the angles $\delta_i (i=X,Y,Z)$. At any time instant t, the actual platform frame can be linked to the ideal platform frame (or for that matter to the navigation frame) with the general transformation:

(7.1.5) $$p^i = C^{p^i}_{p^a_t}\, p^a_t = R^{p^i}_{p^a_{t_o}}\, R^{p^a_{t_o}}_{p^a_t}\, p^a_t \,.$$

where the $R^{p^a_{t_o}}_{p^a_t}$ - transformation relates the actual platform frame at time instant t to that actual platform frame at the starting point and the $R^{p^i}_{p^a_{t_o}}$ - transformation is the initial misalignment transformation.

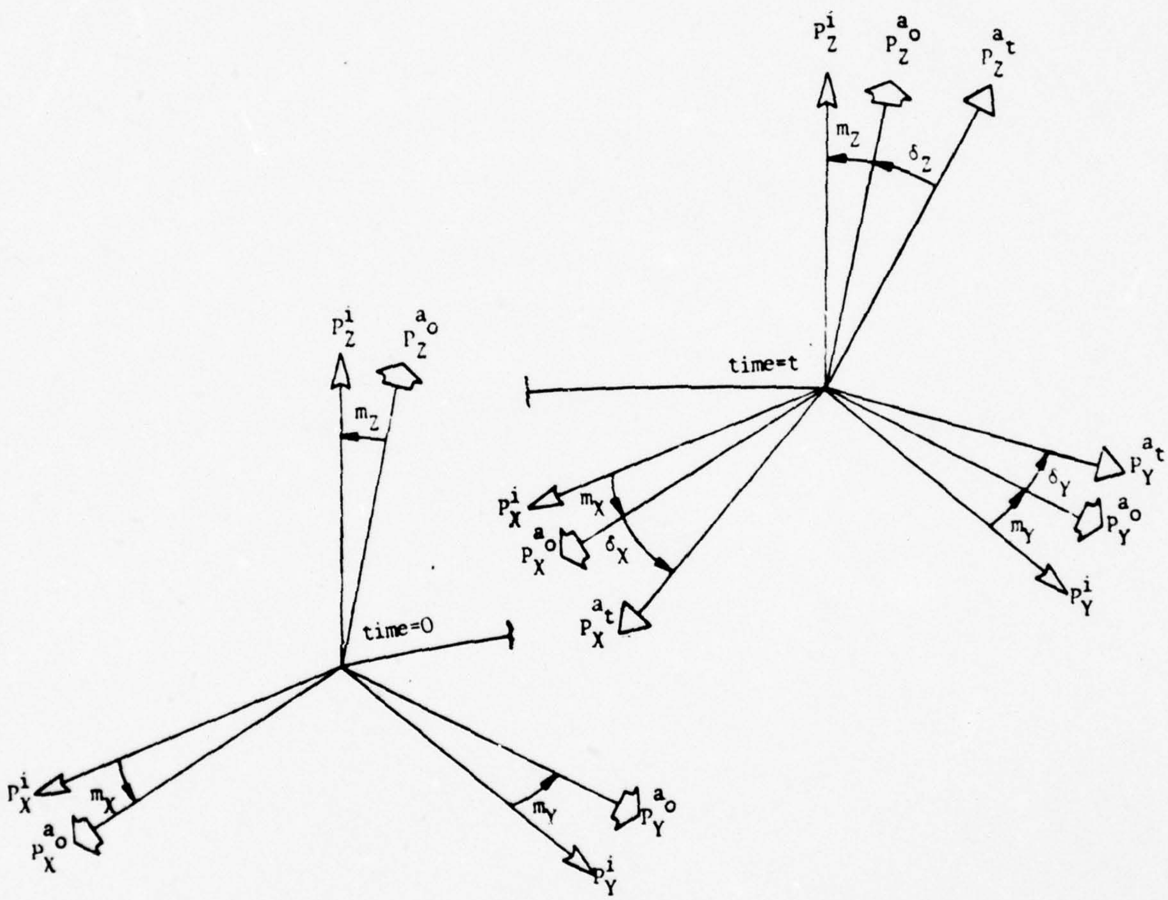


Fig. (10): Platform frames variations.

The $R^{p^i}_{p^a_{t_o}}$ - transformation is a constant matrix transformation and presents no dificulties. According to the given angles definition, we could write(see Fig.(10)):

$$(7.1.6)\qquad R^{p^i}_{p^a_{t_o}} = \begin{bmatrix} 1 & -m_Z & m_Y \\ m_Z & 1 & -m_X \\ -m_Y & m_X & 1 \end{bmatrix}$$

Let us now abstractly denote the $R^{p^a_{t_o}}_{p^a_t}$ - transformation by

$$(7.1.7)\qquad R^{p^a_{t_o}}_{p^a_t} = \begin{bmatrix} 1 & -\delta_Z & \delta_Y \\ \delta_Z & 1 & -\delta_X \\ -\delta_Y & \delta_X & 1 \end{bmatrix}$$

where the elements of the matrix represent small angles or rotations due to gyro errors. Let us therefore analyse them:

a)the gyro frame is constructed by the three spin axes of three on-board mounted gyros. They, generally, instrument a non-orthogonal or quasi-orthogonal frame. Consequently, the transformation between the quasi-orthogonal gyro frame and the actual platform frame reads:

$$(7.1.8)\qquad C^{p^a_t}_{g} = \begin{bmatrix} 0 & -\phi_{YZ} & \phi_{ZY} \\ \phi_{XZ} & 0 & -\phi_{ZX} \\ -\phi_{XY} & \phi_{YX} & 0 \end{bmatrix}$$

where the ϕ's represent small misalignment angles and the same comments as in the accelerometer case apply to the gyro non-orthogonality.

b)now, we demand from the gyros to command the platform with the earth's rotation, but since the three gyros have generally different scale factor uncertainty, then the signal for the respective rotation is falsified. The gyro scale factor uncertainty matrix is expressed as

$$(7.1.9)\qquad U = \begin{bmatrix} U_X & 0 & 0 \\ 0 & U_Y & 0 \\ 0 & 0 & U_Z \end{bmatrix}$$

where U_X, U_Y and U_Z represent the X,Y and Z gyro scale factor uncertainty respectively

c) taking into account eqs (7.1.7), (7.1.8) and (7.1.9), we find the error angle δ to be expressed as:

$$(7.1.10)\qquad \begin{bmatrix} \delta_X \\ \delta_Y \\ \delta_Z \end{bmatrix} = \begin{bmatrix} U_X-\phi_{YZ}+\phi_{ZY} \\ U_Y+\phi_{XZ}-\phi_{ZX} \\ U_Z-\phi_{XY}+\phi_{YX} \end{bmatrix} \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix}$$

where $\omega_X, \omega_Y, \omega_Z$ the earth's rotation components. From the last equation, it is clearly seen that the gyros channel a signal for the earth's rotation to the actual platform frame plus an error rotation due to the gyro non-orthogonality and scale factor uncertainty. This error signal is proportional to the applied rotation, in our case the earth's rotation. Consequently, the $R^{P^a}_{P^a_t}$ to-transformation can be analytically written as

$$(7.1.11)\qquad R^{P^a_{t_o}}_{P^a_t} = \begin{bmatrix} 1 & -(U_Z-\phi_{XY}+\phi_{YX})\omega_Z & (U_Y+\phi_{XZ}-\phi_{ZX})\omega_Y \\ (U_Z-\phi_{XY}+\phi_{YX})\omega_Z & 1 & -(U_X-\phi_{YZ}+\phi_{ZY})\omega_X \\ -(U_Y+\phi_{XZ}-\phi_{ZX})\omega_Y & (U_X-\phi_{YZ}+\phi_{ZY})\omega_X & 1 \end{bmatrix}$$

The off-diagonal terms of the above matrix are time dependent quantities and describe that as the gyro frame drifts changing its angles of non-orthogonality, then the attitude of the actual platform frame is affected. In case in which the gyro frame is orthogonal, it is not drifting, has no scale factor uncertainty and in absence of initial misalignment, then the actual platform frame is nothing else but the ideal platform frame. It is therefore seen that the time increasing gyro drift causes the above off-diagonal terms (those inside the parentheses) to exist.

Taking into account eqs (7.1.5), (7.1.6) and (7.1.11), we find the general transformation taking the accelerometer signal from the misaligned, non-orthogonal accelerometer frame to the navigation earth-linked frame to read:

$$
\begin{bmatrix} A_X^N \\ A_Y^N \\ A_Z^N \end{bmatrix}
=
\left[\begin{array}{c|c|c}
\begin{matrix}
1-(U_Z-\phi_{XY}+\phi_{YX})m_Z\omega_Z \\
+m_Y\omega_Y(-U_Z-\phi_{XZ}+\phi_{ZX}) \\
-\varepsilon_{YZ}\omega_Z(U_Z-\phi_{XY}+\phi_{YX})-m_Z\varepsilon_{YZ} \\
+m_Y\omega_X\varepsilon_{XZ}(U_X-\phi_{YZ}+\phi_{ZY}) \\
-\varepsilon_{ZY}\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
-\varepsilon_{ZY}m_Z\omega_X(U_X-\phi_{YZ}+\phi_{ZY})-m_Y\varepsilon_{ZY}
\end{matrix}
&
\begin{matrix}
-\varepsilon_{XZ}+\varepsilon_{XZ}m_Z\omega_Z(U_Z+\phi_{YX}-\phi_{XY}) \\
-m_Y\omega_Y\varepsilon_{XZ}(-U_Y-\phi_{XZ}+\phi_{ZX}) \\
-\omega_Z(U_Z-\phi_{XY}+\phi_{YX})-m_Z \\
+m_Y\omega_X(U_X-\phi_{YZ}+\phi_{ZY}) \\
+\varepsilon_{ZX}\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
+\varepsilon_{ZX}m_Z\omega_X(U_X-\phi_{YZ}+\phi_{ZY})+\varepsilon_{ZX}m_Y
\end{matrix}
&
\begin{matrix}
\varepsilon_{XZ}-\varepsilon_{XY}m_Z\omega_Z(U_Z-\phi_{XY}+\phi_{YX}) \\
+\varepsilon_{XY}m_Y\omega_Y(-U_Y-\phi_{XZ}+\phi_{ZX}) \\
+\varepsilon_{YX}\omega_Z(U_Z-\phi_{XY}+\phi_{YX})+\varepsilon_{YX}m_Z \\
-\varepsilon_{YX}m_Y\omega_X(U_X-\phi_{YZ}+\phi_{ZY}) \\
+\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
+m_Z\omega_X(U_X-\phi_{YZ}+\phi_{ZY})+m_Y
\end{matrix}
\\ \hline
\begin{matrix}
m_Z+\omega_Z(U_Z-\phi_{XY}+\phi_{YX}) \\
+\omega_Z(U_Y+\phi_{XZ}-\phi_{ZX}) \\
+\varepsilon_{YZ}m_Z\omega_Z(-U_Z+\phi_{XY}-\phi_{YX})+\varepsilon_{YZ} \\
-\varepsilon_{YZ}m_X\omega_X(U_X-\phi_{YZ}+\phi_{ZY}) \\
-\varepsilon_{ZY}m_Z\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
+\varepsilon_{ZY}\omega_Y(U_X-\phi_{YZ}+\phi_{ZY})+\varepsilon_{ZY}m_X
\end{matrix}
&
\begin{matrix}
1-m_Z\omega_Z(U_Z-\phi_{XY}+\phi_{YX}) \\
-m_X\omega_X(U_X-\phi_{YZ}+\phi_{ZY}) \\
-\varepsilon_{XZ}m_Z-\varepsilon_{XZ}\omega_Z(U_Z-\phi_{XY}+\phi_{YX}) \\
-\varepsilon_{XZ}m_Y\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
+\varepsilon_{ZX}m_Z\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
-\varepsilon_{ZX}\omega_X(U_X-\phi_{YZ}+\phi_{ZY})-\varepsilon_{ZX}m_X
\end{matrix}
&
\begin{matrix}
\varepsilon_{XY}m_Z+\varepsilon_{XY}\omega_Z(U_Z-\phi_{XY}+\phi_{YX}) \\
+\varepsilon_{XY}m_X\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
+\varepsilon_{YX}m_Z\omega_Z(U_Z-\phi_{XY}+\phi_{YX})-\varepsilon_{YX} \\
+\varepsilon_{YX}m_X\omega_X(U_X-\phi_{YZ}+\phi_{ZY}) \\
+m_Z\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
-\omega_X(U_X-\phi_{YZ}+\phi_{ZY})-m_X
\end{matrix}
\\ \hline
\begin{matrix}
-m_Y+m_X\omega_Z(U_Z-\phi_{XY}+\phi_{YX}) \\
-\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
+\varepsilon_{YZ}m_Y\omega_Z(U_Z-\phi_{XY}+\phi_{YX})+\varepsilon_{YZ}m_X \\
+\varepsilon_{YZ}\omega_X(U_X-\phi_{YZ}+\phi_{ZY}) \\
+\varepsilon_{ZY}m_Y\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
+\varepsilon_{ZY}m_X\omega_X(U_X-\phi_{YZ}+\phi_{ZY})-\varepsilon_{ZY}
\end{matrix}
&
\begin{matrix}
\varepsilon_{XZ}m_Y-\varepsilon_{XZ}m_X\omega_Z(U_Z-\phi_{XY}+\phi_{YX}) \\
+\varepsilon_{XZ}\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
+m_Y\omega_Z(U_Z-\phi_{XY}+\phi_{YX})+m_X \\
+\omega_X(U_X-\phi_{YZ}+\phi_{ZY}) \\
-\varepsilon_{ZX}m_Y\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
-\varepsilon_{ZX}m_X\omega_X(U_X-\phi_{YZ}+\phi_{ZY})+
\end{matrix}
&
\begin{matrix}
1-\varepsilon_{XY}m_Y+\varepsilon_{XY}m_X\omega_Z(U_Z-\phi_{XY}+\phi_{YX}) \\
-\varepsilon_{XY}\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
-\varepsilon_{YX}m_Y\omega_Z(U_Z-\phi_{XY}+\phi_{YX})-\varepsilon_{YX}m_X \\
-\varepsilon_{YX}\omega_X(U_X-\phi_{YZ}+\phi_{ZY}) \\
-m_Y\omega_Y(U_Y+\phi_{XZ}-\phi_{ZX}) \\
-m_Y\omega_X(U_X-\phi_{YZ}+\phi_{ZY})
\end{matrix}
\end{array}\right]
\begin{bmatrix} A_X^a \\ A_Y^a \\ A_Z^a \end{bmatrix}
\qquad (7.1.12)
$$

Now, since the apparent acceleration components are measured by three actual accelerometers, it is logical to assume an overall accelerometer error model. Taking into account that each accelerometer has its own scale factor uncertainty, bias and random uncertainty, a general accelerometer error model could be expressed as (Britting,1971, Denhard, 1977):

$$\begin{bmatrix} A_X^a \\ A_Y^a \\ A_Z^a \end{bmatrix} = \begin{bmatrix} 1+a_X & 0 & 0 \\ 0 & 1+a_Y & 0 \\ 0 & 0 & 1+a_Z \end{bmatrix} \begin{bmatrix} A_X^{in} \\ A_Y^{in} \\ A_Z^{in} \end{bmatrix} + \begin{bmatrix} b_X^a \\ b_Y^a \\ b_Z^a \end{bmatrix} + \begin{bmatrix} u_X^a \\ u_Y^a \\ u_Z^a \end{bmatrix} \tag{7.1.13}$$

where A_i^a (i=X,Y,Z) indicates the apparent acceleration output signal of the quasi-orthogonal accelerometer frame
A_i^{in}(i=X,Y,Z) indicates the apparent acceleration as an input in the accelerometer frame
a_i (i=X,Y,Z) the accelerometer scale factor uncertainty
b_i^a (i=X,Y,Z) the accelerometer bias and
u_i^a (i=X,Y,Z) the accelerometer random uncertainty.

As it is easily seen, eq.(7.1.13) is an affine transformation. Needless to say that the left hand side of this equation is the acceleration signal to be transformed to the navigation frame, as per eq.(7.1.12), in order to be used for the simulation studies.

7.2 Gradiometer error studies

In our inertial navigation platform, the spherical gradiometer employed has been developed and tested in M.I.T. Each instrument has the capability of measuring two independent gravity gradients and therefore, three of them could furnish the whole gravity gradient tensor plus a redundant gradient indicating accuracy. As it is intuitively understood, the measuring process is quite complicated due to the inherent electronics, but in principle the following fundamental ideas are very helpful:

a)Going into the very beginning of the gradiometer unit, what is really sensed and measured is nothing else but rotations of the float with respect to the stabilised housing. These rotations are sensed by a set of electronic axes and applied back to restore the initial float attitude. The measurements of these rotations represent measurements of gravity gradients. Taking into account the most general case in which the electronic frame is non-orthogonal and is slightly misaligned with respect to the float frame, then these effects have an error influence on the measurements of the gravity gradients which must be anyway compensated.
b)The float frame has a certain prescribed orientation with respect to the axes of principal moments of inertia. But due to various reasons, e.g. inability in locating for perfect the axes of principal moments of inertia, the float frame is thus considered to be slightly misaligned with respect to the ideal float frame.
c)As all instrumental packages so the gradiometer one has its own instrumental axes along of which the gravity gradients are measured. Our gradiometer package frame is the ideal float frame into which gravity gradients measured by a non-orthogonal electronic frame must be finally transformed.
It is now clear that on each float four different sets of coordinate frames exist. These frames are:

1. Principal moments of inertia coordinate frame(P_i)
2. Gradiometer measurement unit coordinate frame(G_i)
3. Actual float frame(F_i) and
4. Electronic frame(E_i)

We have to remark that the P_i and G_i frames are invariant from the gradiometer configuration, but the frames F_i and E_i do depend on that as per Fig.(11).

In order to get the expressions of the gravity gradients referred to the G_i-frame, we proceed as follows:
a)from the general gradiometer torques equations find their respective ones expressed in the G_i, F_i, E_i-frames, for the torque equations are given in the P_i-frame.
b)find the relations between torques in the electronic frame and gravity gradients expressed in the gradiometer measurement unit frame as

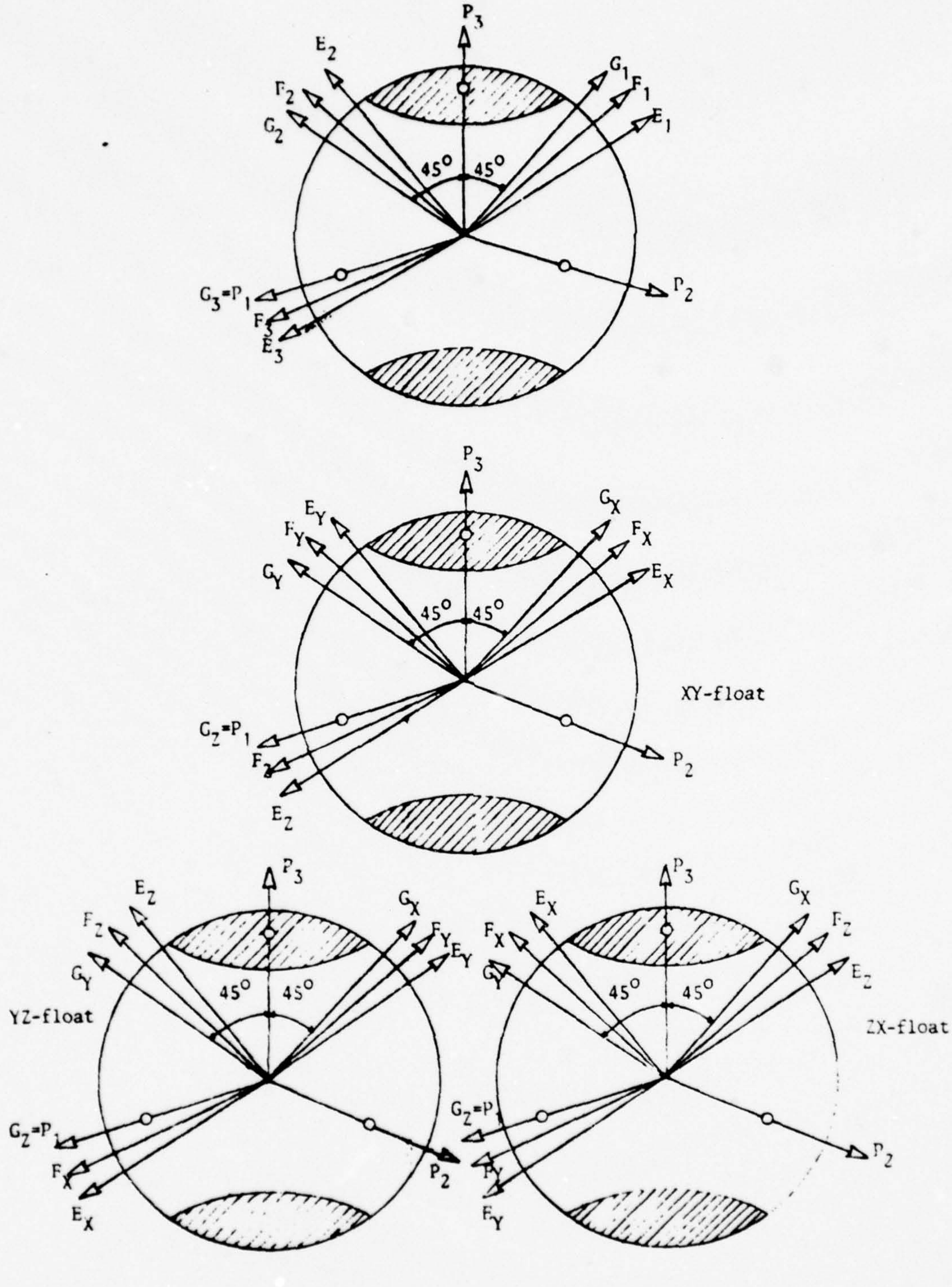


Fig.(11): Spherical gradiometer geometry and reference systems

well as instrumental non-orthogonalities and misalignment angles.
c)find the expressions for the gravity gradients in the P_i-frame with respect to the torques in the same frame.
d)get the final relationship between gravity gradients in the electronic frame and those referred to the gradiometer measurement unit frame.

As soon as the steps a)-d) have been carried out and the gravity gradients in the G_i-frame have been got, then we have to consider a general misorientation of that frame with respect to the inertial one to which the gravity gradients finally refer. Then, the gradient tensor after this very lengthy procedure could be used in the final equation of inertial navigation if and only if a special coordinate transformation is applied to "switch" the gradients from the inertial frame into the operational earth-fixed navigation frame.

Strictly speaking, the mentioned navigation frame cannot be used since the coordinate differences referred to it cannot be integrated. The reason for that is that such a frame is affected by time-like misclosures due to polar motion. These misclosures have been computed, but in terrestrial navigation applications are to be safely neglected(Doukakis,1978).

As it is seen from Fig.(11), the moments or torques measured along P_1 and P_2 principal moments of inertia axes are given(Trageser,1975):

$$M_1 = -\Delta I g_{P_2P_3}$$

$$(7.2.1)\quad M_2 = \Delta I g_{P_1P_3}$$

$$\Delta I = I_{P_2P_2} - I_{P_3P_3}$$

Eqs(7.2.1) will now be transformed into the gradiometer measurement unit axes. Let us first consider the XY-float configuration and particularly the (P_2P_3)-plane(the same analysis is applied in all configurations by a simple permutation on the indices). The axes are depicted in Fig.(12). From elementary plane vector calculus we get

$$(7.2.2)\quad \begin{aligned} \vec{G}_X &= \vec{P}_2\cos45^\circ + \vec{P}_3\sin45^\circ \\ \vec{G}_Y &= -\vec{P}_2\sin45^\circ + \vec{P}_3\cos45^\circ \end{aligned}$$

Representing the gravity gradient field by Γ, we could write

$$(7.2.3)\quad \Gamma = \vec{P}_2\vec{P}_3 g_{P_2P_2} + \vec{P}_3\vec{P}_3 g_{P_3P_3} + \vec{P}_2\vec{P}_3 g_{P_2P_3} + \vec{P}_3\vec{P}_2 g_{P_3P_2}$$

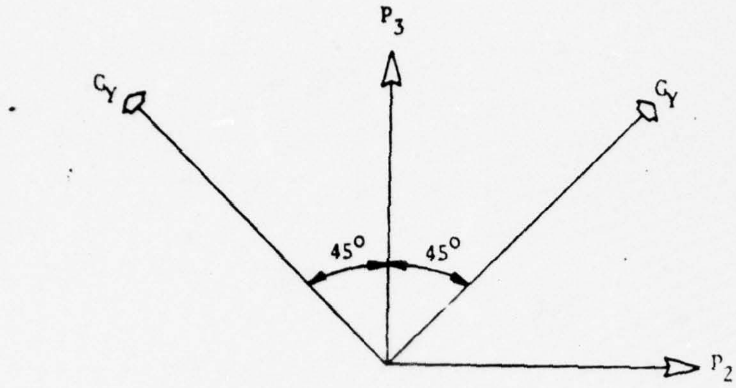


Fig.(12): Axes configuration in the (P_2P_3)-plane of the XY-float.

where the g's represent gravity gradients, Now, each gradient can be written as

(7.2.4) $\quad \vec{g}_{ij} = \vec{i}\cdot(\Gamma\cdot\vec{j})$

and thus we get

$$\frac{1}{2}(\overset{G}{g}_{YY}-\overset{G}{g}_{XX}) = \frac{1}{2}(\vec{G}_Y(\Gamma\cdot\vec{G}_Y)-\vec{G}_X(\Gamma\cdot\vec{G}_X)) = \frac{1}{2}(\overset{P}{g}_{P_3P_3}-\overset{P}{g}_{P_2P_2})\cos 2\cdot 45^\circ - \overset{P}{g}_{P_3P_2}\sin 2\cdot 45^\circ$$

(7.2.5) $$= -\overset{P}{g}_{P_3P_2}$$

where the letter over the gravity gradients indicates the frame which they refer to.

In view of eqs(7.2.1) and (7.2.5) as well as the XY-float configuration schematic, we get

$$\overset{G}{M}_Z = \frac{\Delta I}{2}(g_{YY}-g_{XX})$$

where it is understood that the gravity gradients refer to the same coordinate frame as the measured moments, then the gravity gradient superscript is dropped for simplicity. If eq(7.2.4) is applied to the other planes, then we can get:

$$\overset{G}{M}_X = \frac{\Delta I}{2}(g_{ZX}+g_{ZY})$$

$$(7.2.6)\qquad \begin{aligned} {}^{G}M_Y &= -\frac{\Delta I}{2}(g_{ZX}+g_{ZY}) \\ {}^{G}M_Z &= \frac{\Delta I}{2}(g_{YY}+g_{XX}) \end{aligned} \qquad \text{XY-float}$$

By a simple permutation of the indices, we obtain for the YZ and ZX-float configurations the following:

$$(7.2.7)\qquad \begin{aligned} {}^{G}M_X &= \frac{\Delta I}{2}(g_{ZZ}-g_{YY}) \\ {}^{G}M_Y &= \frac{\Delta I}{2}(g_{XY}+g_{XZ}) \\ {}^{G}M_Z &= -\frac{\Delta I}{2}(g_{XY}+g_{XZ}) \end{aligned} \qquad \text{YZ-float}$$

$$\begin{aligned} {}^{G}M_X &= -\frac{\Delta I}{2}(g_{YZ}+g_{YX}) \\ {}^{G}M_Y &= -\frac{\Delta I}{2}(g_{YZ}+g_{YX}) \\ {}^{G}M_Z &= \frac{\Delta I}{2}(g_{YZ}+g_{YX}) \end{aligned} \qquad \text{ZX-float}$$

Eqs(7.2.7) express themoments coordinatized in the gradiometer measurement unit frame and hold as they stand for the actual float frame F_i as well as the electronic frame E_i changing only the superscript G by F and E respectively.

We have now to find the relation between the torques measured in the electronic frame, being the gradiometer sensor for gravity gradients, and the gravity gradients coordinatized in the gradiometer measurement unit frame taking into account instrumental non-orthogonality and misalignment. First, let us consider the case in which the electronic frame is a non-orthogonal one and also misaligned with respect to the actual float frame(see Fig.(13)). As we have already explained in the accelerometer studies, the matrix which transforms the moments of the electronic frame to those of the float frame is exactly as per eq.(7.1.4). Therefore, we write

$$(7.2.8)\qquad \begin{aligned} {}^{F}M_X &= {}^{E}M_X-\theta_{YZ}\,{}^{E}M_Y+\theta_{ZY}\,{}^{E}M_Z \\ {}^{F}M_Y &= \theta_{XZ}\,{}^{E}M_X+{}^{E}M_Y-\theta_{ZX}\,{}^{E}M_Z \\ {}^{F}M_Z &= -\theta_{XY}\,{}^{E}M_X+\theta_{YX}\,{}^{E}M_Y+{}^{E}M_Z \end{aligned}$$

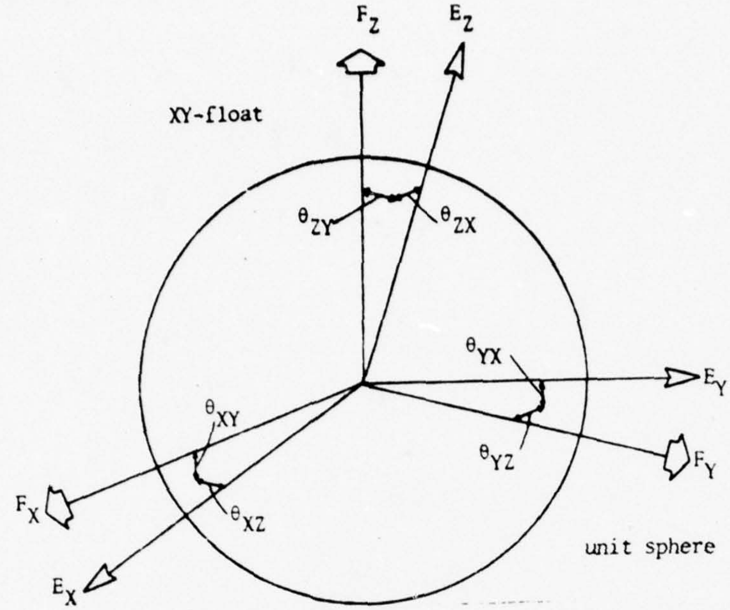


Fig.(13): Relation between actual float and electronic frames.

Solution of eqs(7.2.8) with respect to the electronic torques, disregarding products of small angles, gives

$$
\begin{aligned}
{}^{E}M_X &= {}^{F}M_X + \theta_{XZ}\,{}^{F}M_Y - \theta_{XY}\,{}^{F}M_Z \\
{}^{E}M_Y &= -\theta_{YZ}\,{}^{F}M_X + {}^{F}M_Y + \theta_{YX}\,{}^{F}M_Z \\
{}^{E}M_Z &= \theta_{ZY}\,{}^{F}M_X - \theta_{ZX}\,{}^{F}M_Y + {}^{F}M_Z
\end{aligned}
\tag{7.2.9}
$$

and since the electronic torques are equal in magnitude and opposite in sign to the torques caused by the gravity field, we get

$$
\begin{aligned}
{}^{E}M_X &= -{}^{F}\bar{M}_X - \theta_{XZ}\,{}^{F}\bar{M}_Y + \theta_{XY}\,{}^{F}\bar{M}_Z \\
{}^{E}M_Y &= \theta_{YZ}\,{}^{F}\bar{M}_X - {}^{F}\bar{M}_Y - \theta_{YX}\,{}^{F}\bar{M}_Z \\
{}^{E}M_Z &= -\theta_{ZY}\,{}^{F}\bar{M}_X + \theta_{ZX}\,{}^{F}\bar{M}_Y - {}^{F}\bar{M}_Z
\end{aligned}
\tag{7.2.10}
$$

where the bar over the torques indicates that these torques come from the physical entity of gravity.

Now, let us transform the gravity gradients which refer to the gradiometer measurement unit frame into those referred to the actual float frame.

Taking into account that the float frame is misaligned with

ψ_1 misalignment angle about the G_X-axis

ψ_2 " " " " G_Y- "

ψ_3 " " " " G_Z- "

then, the gravity gradients sensed by the XY-float configuration read

$$
\begin{aligned}
g^F_{XX} &= g^G_{XX} - 2g^G_{XY}\psi_2 + 2g^G_{XY}\psi_3 \\
g^F_{YY} &= g^G_{YY} + 2g^G_{YZ}\psi_1 - 2g^G_{XY}\psi_3 \\
g^F_{ZX} &= g^G_{ZX} - g^G_{YX}\psi_1 + (g^G_{XX} - g^G_{ZZ})\psi_2 + g^G_{ZY}\psi_3 \\
g^F_{ZY} &= g^G_{ZY} + (g^G_{ZZ} - g^G_{YY})\psi_1 + g^G_{XY}\psi_2 - g^G_{ZX}\psi_3
\end{aligned}
\tag{7.2.11}
$$

We rewrite now the first three equations from eqs(7.2.7) in the actual float frame as

$$
\begin{aligned}
M^F_X &= \frac{\Delta I}{2}(g^F_{ZX} + g^F_{ZY}) \\
M^F_Y &= -\frac{\Delta I}{2}(g^F_{ZX} + g^F_{ZY}) \\
M^F_Z &= \frac{\Delta I}{2}(g^F_{YY} - g^F_{XX})
\end{aligned}
\tag{7.2.12a}
$$

Solution of the above equations with respect to the gravity gradients gives

$$
\begin{aligned}
g^F_{ZX} + g^F_{ZY} &= \frac{\Delta I}{2} M^F_X \\
g^F_{ZX} + g^F_{ZY} &= -\frac{\Delta I}{2} M^F_Y \\
g^F_{YY} - g^F_{XX} &= \frac{\Delta I}{2} M^F_Z
\end{aligned}
\tag{7.2.12b}
$$

Combining eqs(7.2.10), (7.2.11) and(7.2.12), we see that the X-channel electronic torque assumes the form

$$
\begin{aligned}
M^E_X = \frac{\Delta I}{2}(&-(g^G_{ZX} - g^G_{YX}\psi_1 + (g^G_{XX} - g^G_{ZZ})\psi_2 + g^G_{ZY}\psi_3 + g^G_{ZY} + (g^G_{ZZ} - g^G_{YY})\psi_1 + g^G_{XY}\psi_2 - g^G_{ZX}\psi_3 - \psi_{XZ}(g^G_{ZX} - g^G_{YX}\psi_1 \ldots \\
&+ (g^G_{XX} - g^G_{ZZ})\psi_2 + g^G_{ZY} + (g^G_{ZZ} - g^G_{YY})\psi_1 + g^G_{XY}\psi_2 - g^G_{ZX}\psi_3 - \psi_{XY}(g^G_{YY} - g^G_{XX} + 2g^G_{YZ}\psi_1 + 2g^G_{XY}\psi_2 - 4g^G_{XY}\psi_3 + g^G_{ZY}\psi_3))
\end{aligned}
$$

After some manipulation, we obtain the electronic torques for all three float configurations to read:

$$\begin{aligned}
{}^{E}M^{XY}_{X} &= \frac{\Delta I}{2}\Big(-(g^{G}_{ZX}+g^{G}_{ZY})+(g^{G}_{YY}-g^{G}_{ZZ}+g^{G}_{YX})\psi_1+(g^{G}_{ZZ}-g^{G}_{XX}+g^{G}_{XY})\psi_2+(g^{G}_{ZX}-g^{G}_{ZY})\psi_3+(g^{G}_{ZX}+g^{G}_{ZY})\theta_{XZ}+\\
&\quad +(g^{G}_{YY}-g^{G}_{XX})\theta_{XY}\Big)\\
{}^{E}M^{XY}_{Y} &= \frac{\Delta I}{2}\Big((g^{G}_{ZX}+g^{G}_{ZY})+(g^{G}_{ZZ}-g^{G}_{YY}-g^{G}_{YX})\psi_1+(g^{G}_{XX}-g^{G}_{ZZ}-g^{G}_{XY})\psi_2+(g^{G}_{ZY}-g^{G}_{ZX})\psi_3+(g^{G}_{ZX}-g^{G}_{ZY})\theta_{YZ}-\\
&\quad -(g^{G}_{YY}-g^{G}_{XX})\theta_{YX}\Big)\\
{}^{E}M^{XY}_{Z} &= \frac{\Delta I}{2}\Big((g^{G}_{XX}-g^{G}_{YY})-2g^{G}_{YZ}\psi_1-2g^{G}_{XZ}\psi_2+4g^{G}_{XY}\psi_3-(g^{G}_{ZX}+g^{G}_{ZY})\theta_{ZY}-(g^{G}_{ZX}+g^{G}_{ZY})\theta_{ZX}\Big)\\
{}^{E}M^{YZ}_{X} &= \frac{\Delta I}{2}\Big((g^{G}_{YY}-g^{G}_{ZZ})+4g^{G}_{YZ}\psi_1-2g^{G}_{ZX}\psi_2-2g^{G}_{YX}\psi_3-(g^{G}_{XY}+g^{G}_{XZ})\theta_{XZ}-(g^{G}_{XY}+g^{G}_{XZ})\theta_{XY}\Big)\\
{}^{E}M^{YZ}_{Y} &= \frac{\Delta I}{2}\Big(-(g^{G}_{XY}+g^{G}_{XZ})+(g^{G}_{XY}-g^{G}_{XZ})\psi_1+(g^{G}_{ZZ}-g^{G}_{XX}+g^{G}_{ZY})\psi_2+(g^{G}_{XX}-g^{G}_{YY}+g^{G}_{YZ})\psi_3+(g^{G}_{ZZ}-g^{G}_{YY})\theta_{YZ}+\\
&\quad +(g^{G}_{XY}+g^{G}_{XZ})\theta_{YX}\Big)\\
{}^{E}M^{YZ}_{Z} &= \frac{\Delta I}{2}\Big((g^{G}_{XY}+g^{G}_{XZ})+(g^{G}_{XZ}-g^{G}_{XY})\psi_1+(g^{G}_{XX}-g^{G}_{YZ}-g^{G}_{ZZ})\psi_2+(g^{G}_{YY}-g^{G}_{XX}-g^{G}_{YZ})\psi_3-(g^{G}_{ZZ}-g^{G}_{YY})\theta_{ZY}+\\
&\quad +(g^{G}_{XY}+g^{G}_{XZ})\theta_{ZX}\Big)\\
{}^{E}M^{ZX}_{X} &= \frac{\Delta I}{2}\Big((g^{G}_{YZ}+g^{G}_{YX})+(g^{G}_{ZZ}-g^{G}_{YY}-g^{G}_{ZX})\psi_1+(g^{G}_{YX}-g^{G}_{YZ})\psi_2+(g^{G}_{YY}-g^{G}_{XX}-g^{G}_{XZ})\psi_3-(g^{G}_{XX}-g^{G}_{ZZ})\theta_{XZ}+\\
&\quad +(g^{G}_{YZ}+g^{G}_{YX})\theta_{XY}\Big)\\
{}^{E}M^{ZX}_{Y} &= \frac{\Delta I}{2}\Big((g^{G}_{ZZ}-g^{G}_{XX})-2g^{G}_{ZY}\psi_1+4g^{G}_{ZX}\psi_2-2g^{G}_{XY}\psi_3-(g^{G}_{YX}+g^{G}_{YZ})\theta_{YZ}-(g^{G}_{XY}+g^{G}_{YZ})\theta_{YX}\Big)\\
{}^{E}M^{ZX}_{Z} &= \frac{\Delta I}{2}\Big(-(g^{G}_{YX}+g^{G}_{YZ})+(g^{G}_{YY}-g^{G}_{ZZ}+g^{G}_{ZX})\psi_1+(g^{G}_{YZ}-g^{G}_{YX})\psi_2+(g^{G}_{XX}-g^{G}_{YY}+g^{G}_{XZ})\psi_3+(g^{G}_{YX}+g^{G}_{YZ})\theta_{ZY}+\\
&\quad +(g^{G}_{XX}-g^{G}_{ZZ})\theta_{ZX}\Big)
\end{aligned}\tag{7.2.13}$$

Now, eqs(7.2.7) are written in the electronic frame as follows:

$$\frac{2}{\Delta I}\,{}^{E}M^{XY}_{X} = g^{E}_{ZX}+g^{E}_{ZY}$$

$$\frac{2}{\Delta I}\,{}^{E}M^{XY}_{Y} = -g^{E}_{ZX}-g^{E}_{ZY}$$

$$\frac{2}{\Delta I}\,{}^{E}M^{XY}_{Z} = g^{E}_{YY}-g^{E}_{XX}$$

$$\frac{2}{\Delta I}\,{}^{E}M^{YZ}_{X} = g^{E}_{ZZ}-g^{E}_{YY}$$

$$\begin{aligned}
&(7.2.14) \quad \frac{2}{\Delta I}\overset{E}{M}{}^{YZ}_{Y} = \overset{E}{g}_{XY}+\overset{E}{g}_{XZ}\\
&\frac{2}{\Delta I}\overset{E}{M}{}^{YZ}_{Z} = -\overset{E}{g}_{XY}-\overset{E}{g}_{XZ}\\
&\frac{2}{\Delta I}\overset{E}{M}{}^{ZX}_{X} = -\overset{E}{g}_{YX}-\overset{E}{g}_{YZ}\\
&\frac{2}{\Delta I}\overset{E}{M}{}^{ZX}_{Y} = \overset{E}{g}_{XX}-\overset{E}{g}_{ZZ}\\
&\frac{2}{\Delta I}\overset{E}{M}{}^{ZX}_{Z} = \overset{E}{g}_{YX}+\overset{E}{g}_{YZ}
\end{aligned}$$

Having excluded eqs(7.2.14b,f,i), the rest six equations are then solved in view of the Laplace condition

$$(7.2.15) \quad g_{XX}+g_{YY}+g_{ZZ} = -4\pi k\rho + 2\omega^2$$

where ω the earth's rotation with respect to the inertial space, k the universal gravitational constant and ρ the density of the medium in which the navigation takes place. The solution gives:

$$\begin{aligned}
&\overset{E}{g}_{XX} = -\frac{4}{3}\pi k\rho+\frac{2}{3}\omega^2+\frac{2}{3\Delta I}\left(2\overset{E}{M}{}^{ZX}_{Y}+\overset{E}{M}{}^{YZ}_{X}\right)\\
&\overset{E}{g}_{XY} = -\frac{1}{\Delta I}\left(\overset{E}{M}{}^{XY}_{X}+\overset{E}{M}{}^{YZ}_{Y}+\overset{E}{M}{}^{ZX}_{X}\right)\\
&\overset{E}{g}_{XZ} = \frac{1}{\Delta I}\left(\overset{E}{M}{}^{XY}_{X}+\overset{E}{M}{}^{YZ}_{Y}+\overset{E}{M}{}^{ZX}_{X}\right)\\
(7.2.16)\quad &\overset{E}{g}_{YY} = -\frac{4}{3}\pi k\rho+\frac{2}{3}\omega^2-\frac{2}{3\Delta I}\left(\overset{E}{M}{}^{ZX}_{Y}+2\overset{E}{M}{}^{YZ}_{X}\right)\\
&\overset{E}{g}_{YZ} = \frac{1}{\Delta I}\left(\overset{E}{M}{}^{XY}_{X}-\overset{E}{M}{}^{YZ}_{Y}-\overset{E}{M}{}^{ZX}_{X}\right)\\
&\overset{E}{g}_{ZZ} = -\frac{4}{3}\pi k\rho+\frac{2}{3}\omega^2-\frac{2}{3\Delta I}\left(\overset{E}{M}{}^{ZX}_{Y}-\overset{E}{M}{}^{YZ}_{X}\right)
\end{aligned}$$

If the electronic torques involved in the above equations are substituted by those given in eqs(7.2.13), then after a tedious manipulation we get the relations between gravity gradients in the electronic frame and gradiometer unit frame to read:

$$\begin{aligned}
\overset{E}{g}_{XX} = &-\frac{4}{3}\pi k\rho+\frac{2}{3}\omega^2+\frac{1}{3}\Big(\overset{G}{g}_{ZZ}+\overset{G}{g}_{YY}-2\overset{G}{g}_{XX}+6\overset{G}{g}_{ZX}\psi_2-6\overset{G}{g}_{xy}\psi_3-(2\theta_{YX}+\theta_{XY}+\theta_{XZ}+2\theta_{YZ})\overset{G}{g}_{XY}-(\theta_{XZ}+\theta_{XY})\overset{G}{g}_{XZ}-\\
&-(\theta_{YX}+2\theta_{YZ})\overset{G}{g}_{XZ}\Big)
\end{aligned}$$

$$g^{E}_{XY}=-\frac{1}{2}\left(2g^{G}_{XY}+2g^{G}_{XY}\psi_2-2g^{G}_{YZ}\psi_2-2g^{G}_{XX}\psi_3+2g^{G}_{YY}\psi_3-(\theta_{XY}+\theta_{XZ})g^{G}_{XX}+(\theta_{XZ}-\theta_{YX})g^{G}_{XZ}+(\theta_{YZ}+\theta_{XY})g^{G}_{YZ}-\right.$$
$$\left.-(\theta_{XZ}+\theta_{YZ})g^{G}_{ZZ}\right)$$

$$g^{E}_{XZ}=\frac{1}{2}\left(-2g^{G}_{XZ}+2g^{G}_{XY}\psi_1+2g^{G}_{ZZ}\psi_2-2g^{G}_{XZ}\psi_1-2g^{G}_{XX}\psi_2-(\theta_{XY}+\theta_{XZ})g^{G}_{XX}+(\theta_{XY}+\theta_{YX})g^{G}_{XY}+(\theta_{XY}+\theta_{YZ})g^{G}_{YY}+\right.$$
$$\left.+(\theta_{XY}+\theta_{XZ})g^{G}_{YZ}+(\theta_{YZ}-\theta_{XZ})g^{G}_{ZZ}\right) \quad (7.2.17)$$

$$g^{E}_{YY}=-\frac{4}{3}\pi k\rho+\frac{2}{3}\omega^2-\frac{1}{3}\left(-g^{G}_{XX}-g^{G}_{ZZ}+2g^{G}_{YY}+6g^{G}_{ZY}\psi_1-6g^{G}_{XY}\psi_3-(2\theta_{XZ}+\theta_{YZ}+\theta_{YX}+2\theta_{XY})g^{G}_{XY}-(\theta_{YZ}+\theta_{YX})g^{G}_{YZ}-\right.$$
$$\left.-(2\theta_{XZ}+2\theta_{XY})g^{G}_{XZ}\right)$$

$$g^{E}_{YZ}=\frac{1}{2}\left(-2g^{G}_{ZY}+2g^{G}_{YY}\psi_1-2g^{G}_{ZZ}\psi_1+2g^{G}_{XZ}\psi_1+2g^{G}_{ZX}\psi_3-2g^{G}_{ZY}\psi_3-(\theta_{YX}-\theta_{XZ})g^{G}_{XX}-(\theta_{YX}+\theta_{XY})g^{G}_{XY}+(\theta_{XZ}-\theta_{YX})g^{G}_{XZ}+\right.$$
$$\left.+(\theta_{YZ}+\theta_{XY})g^{G}_{YY}+(\theta_{XZ}-\theta_{XY})g^{G}_{YZ}+(\theta_{XZ}-\theta_{YZ})g^{G}_{ZZ}\right)$$

$$g^{E}_{ZZ}=-\frac{4}{3}\pi k\rho+\frac{2}{3}\omega^2-\frac{1}{3}\left(2g^{G}_{ZZ}-g^{G}_{XX}-g^{G}_{YY}-6g^{G}_{ZY}\psi_1+6g^{G}_{XZ}\psi_2+(\theta_{XZ}-\theta_{YX}-\theta_{YZ}+\theta_{XY})g^{G}_{XY}+(\theta_{XZ}+\theta_{XY})g^{G}_{XZ}-\right.$$
$$\left.-(\theta_{YX}+\theta_{YZ})g^{G}_{YZ}\right)$$

Eqs(7.2.17) can be written in the concise matrix form of eq.(7.2.18).

Now, the G-frame is considered to be misaligned with respect to the inertial frame. Denoting by superscript I those gravity gradients referred to the inertial frame, then we write

$$g^{I}_{ij}=\sum_{k=1}^{3}\sum_{l=1}^{3}a_{ik}a_{jl}g^{G}_{kl} \quad (7.2.19)$$

where i,j=X,Y,Z and the a's represent the element of the transformation matrix between the two frames. Analytically, the gradients are written:

$$g^{I}_{XX}=a_{XX}a_{XX}g_{XX}+a_{XX}a_{XY}g_{XY}+a_{XX}a_{XZ}g_{XZ}+a_{XY}a_{XX}g_{YX}+a_{XY}a_{XY}g_{YY}+a_{XY}a_{XZ}g_{YZ}+a_{XZ}a_{XX}g_{ZX}+a_{XZ}a_{XY}g_{ZY}+$$
$$+a_{XZ}a_{XZ}g_{ZZ}$$

$$g^{I}_{XY}=a_{XX}a_{YX}g_{XX}+a_{XX}a_{YY}g_{XY}+a_{XX}a_{YZ}g_{XZ}+a_{XY}a_{YX}g_{YX}+a_{XY}a_{YY}g_{YY}+a_{XY}a_{YZ}g_{YZ}+a_{XZ}a_{YX}g_{ZX}+a_{XZ}a_{YY}g_{ZY}+$$
$$+a_{XZ}a_{YZ}g_{ZZ}$$

$$g^{I}_{XZ}=a_{XX}a_{ZX}g_{XX}+a_{XX}a_{ZY}g_{XY}+a_{XX}a_{ZZ}g_{XZ}+a_{XY}a_{ZX}g_{YX}+a_{XY}a_{ZY}g_{YY}+a_{XY}a_{ZZ}g_{YZ}+a_{XZ}a_{ZX}g_{ZX}+a_{XZ}a_{ZY}g_{ZY}+$$
$$+a_{XZ}a_{ZZ}g_{ZZ} \quad (7.2.20)$$

$$
\begin{bmatrix} g^{E}_{XX} \\ g^{E}_{XY} \\ g^{E}_{XZ} \\ g^{E}_{YY} \\ g^{E}_{YZ} \\ g^{E}_{ZZ} \end{bmatrix}
=
\begin{bmatrix} -\frac{8}{3}\pi k\rho + \frac{4}{3}\omega^2 \\ 0 \\ 0 \\ -\frac{8}{3}\pi k\rho + \frac{4}{3}\omega^2 \\ 0 \\ -\frac{8}{3}\pi k\rho + \frac{4}{3}\omega^2 \end{bmatrix}
+
\left[\begin{array}{c|c|c|c|c|c}
-1 & -2\psi_3 - \frac{1}{3}(2\theta_{YX}+\theta_{XY}+\theta_{XZ}+2\theta_{YZ}) & +2\psi_2 - \frac{1}{3}(\theta_{XZ}+\theta_{XY}) & 0 & -\frac{1}{3}(\theta_{YX}+2\theta_{YZ}) & 0 \\ \hline
\psi_3 + \frac{1}{2}(\theta_{XY}+\theta_{XZ}) & -(1+\psi_2) & -\frac{1}{2}(\theta_{XZ}-\theta_{YX}) & -\psi_3 - \frac{1}{2}(\theta_{YZ}+\theta_{XY}) & \psi_2 - \frac{1}{2}(\theta_{XZ}+\theta_{XY}) & \frac{1}{2}(\theta_{XZ}+\theta_{YZ}) \\ \hline
-\psi_2 - \frac{1}{2}(\theta_{XY}+\theta_{XZ}) & \psi_1 + \frac{1}{2}(\theta_{XY}+\theta_{YX}) & -1-\psi_1+\frac{1}{2}(\theta_{XZ}+\theta_{YX}) & \frac{1}{2}(\theta_{XY}-\theta_{YZ}) & \frac{1}{2}(\theta_{XY}+\theta_{XZ}) & \psi_2+\frac{1}{2}(\theta_{YZ}-\theta_{XZ}) \\ \hline
0 & 2\psi_3 + \frac{1}{3}(2\theta_{XZ}+\theta_{YZ}+\theta_{YX}+2\theta_{XY}) & +\frac{1}{3}(2\theta_{XZ}+2\theta_{XY}) & -1 & -2\psi_1+\frac{1}{3}(\theta_{YZ}+\theta_{YX}) & 0 \\ \hline
\frac{1}{2}(\theta_{XZ}-\theta_{YX}) & -\frac{1}{2}(\theta_{YX}+\theta_{XY}) & +\psi_1 + \frac{1}{2}(\theta_{XZ}-\theta_{YX}) & +\psi_1 + \frac{1}{2}(\theta_{YZ}+\theta_{XY}) & -1+\frac{1}{2}(\theta_{XZ}-\theta_{XY}) & -\psi_1+\frac{1}{2}(\theta_{XZ}-\theta_{YZ}) \\ \hline
0 & -\frac{1}{3}(\theta_{XZ}-\theta_{YX}-\theta_{YZ}+\theta_{XY}) & -\psi_2 - \frac{1}{3}(\theta_{XZ}+\theta_{XY}) & 0 & 2\psi_1+\frac{1}{3}(\theta_{YX}+\theta_{YZ}) & -1
\end{array}\right]
\begin{bmatrix} g^{G}_{XX} \\ g^{G}_{XY} \\ g^{G}_{XZ} \\ g^{G}_{YY} \\ g^{G}_{YZ} \\ g^{G}_{ZZ} \end{bmatrix}
\qquad (7.2.18)
$$

$$\overset{I}{g}_{YY}=a_{YX}a_{YX}g_{XX}+a_{YX}a_{YY}g_{XY}+a_{YX}a_{YZ}g_{XZ}+a_{YY}a_{YX}g_{YX}+a_{YY}a_{YY}g_{YY}+a_{YY}a_{YZ}g_{YZ}+a_{YZ}a_{YX}g_{ZX}+a_{YZ}a_{YY}g_{ZY}+$$
$$+a_{YZ}a_{YZ}g_{ZZ}$$

$$\overset{I}{g}_{YZ}=a_{YX}a_{ZX}g_{XX}+a_{YX}a_{ZY}g_{XY}+a_{YX}a_{ZZ}g_{XZ}+a_{YY}a_{ZX}g_{YX}+a_{YY}a_{ZY}g_{YY}+a_{YY}a_{ZZ}g_{YZ}+a_{YZ}a_{ZX}g_{ZX}+a_{YZ}a_{ZY}g_{ZY}+$$
$$+a_{YZ}a_{ZZ}g_{ZZ}$$

$$\overset{I}{g}_{ZZ}=a_{ZX}a_{ZX}g_{XX}+a_{ZX}a_{ZY}g_{XY}+a_{ZX}a_{ZZ}g_{XZ}+a_{ZY}a_{ZX}g_{YX}+a_{ZY}a_{ZY}g_{YY}+a_{ZY}a_{ZZ}g_{YZ}+a_{ZZ}a_{ZX}g_{ZX}+a_{ZZ}a_{ZY}g_{ZY}+$$
$$+a_{ZZ}a_{ZZ}g_{ZZ}$$

where all gravity gradients at the right hand side refer to the G-frame and the superscript G has been dropped for simplicity. Since the misalignment between the G-frame and the inertial one is in the "small angle" sense, then we can make the approximations:

$$\begin{array}{lll} a_{XX}=1 & a_{YX}=-\xi_Z & a_{ZZ}=\xi_Y \\ a_{XY}=\xi_Z & a_{YY}=1 & a_{ZY}=-\xi_X \\ a_{XZ}=-\xi_Y & a_{YZ}=\xi_X & a_{ZZ}=1 \end{array} \qquad (7.2.21)$$

where the ξ_X, ξ_Y, ξ_Z represent small rotation angles about the G_X, G_Y and G_Z gradiometer axes respectively. Combining eqs(7.2.20) and (7.2.21), we get:

$$\begin{aligned}
\overset{I}{g}_{XX}&=\overset{G}{g}_{XX}-2\xi_Y\overset{G}{g}_{XZ}+2\xi_Z\overset{G}{g}_{XY}\\
\overset{I}{g}_{XY}&=\overset{G}{g}_{XY}+\xi_X\overset{G}{g}_{XZ}-\xi_Y\overset{G}{g}_{ZY}+\xi_Z(\overset{G}{g}_{YY}-\overset{G}{g}_{XX})\\
\overset{I}{g}_{XZ}&=\overset{G}{g}_{XZ}-\xi_X\overset{G}{g}_{XY}+\xi_Y(\overset{G}{g}_{XX}-\overset{G}{g}_{ZZ})+\xi_Z\overset{G}{g}_{YZ}\\
\overset{I}{g}_{YY}&=\overset{G}{g}_{YY}+2\xi_X\overset{G}{g}_{ZY}-2\xi_Z\overset{G}{g}_{XY}\\
\overset{I}{g}_{YZ}&=\overset{G}{g}_{YZ}+\xi_X(\overset{G}{g}_{ZZ}-\overset{G}{g}_{YY})+\xi_Y\overset{G}{g}_{YX}-\xi_Z\overset{G}{g}_{XZ}\\
\overset{I}{g}_{ZZ}&=\overset{G}{g}_{ZZ}-2\xi_X\overset{G}{g}_{YZ}+2\xi_Y\overset{G}{g}_{XZ}
\end{aligned} \qquad (7.2.22a)$$

Eqs(7.2.22a) will now be written in matrix form for further reference as:

$$\begin{bmatrix} g_{XX} \\ g_{XY} \\ g_{XZ} \\ g_{YY} \\ g_{YZ} \\ g_{ZZ} \end{bmatrix} = \begin{bmatrix} 1 & 2\xi_Z & -2\xi_Y & 0 & 0 & 0 \\ -\xi_Z & 1 & \xi_X & \xi_Z & -\xi_Y & 0 \\ \xi_Y & -\xi_X & 1 & 0 & \xi_Z & -\xi_Y \\ 0 & -2\xi_Z & 0 & 1 & 2\xi_X & 0 \\ 0 & \xi_Y & -\xi_Z & -\xi_X & 1 & \xi_X \\ 0 & 0 & 2\xi_Y & 0 & -2\xi_X & 1 \end{bmatrix} \begin{bmatrix} g_{XX} \\ g_{XY} \\ g_{XZ} \\ g_{YY} \\ g_{YZ} \\ g_{ZZ} \end{bmatrix} \tag{7.2.22b}$$

Now, the gravity gradients refer to the inertial frame, but in order to be used as those entering the fundamental equation of inertial navigation, they have to be transformed into the navigation frame. The transformation matrix between the selected inertial frame and the earth-linked navigation frame is parametarized by the true sidereal time θ and the coordinates of the polar motion x and y referred to a specified epoch. This transformation is given(Veis,1962):

$$[N] = \begin{bmatrix} \cos\theta & \sin\theta & x \\ -\sin\theta & \cos\theta & -y \\ -x\cos\theta-y\sin\theta & -x\sin\theta+y\cos\theta & 1 \end{bmatrix} [I]$$

and according to standard literature the gravity gradients are transformed as follows:

$$\begin{bmatrix} g_{XX} & g_{XY} & g_{XZ} \\ g_{YX} & g_{YY} & g_{YZ} \\ g_{ZX} & g_{ZY} & g_{ZZ} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & -x\cos\theta-y\sin\theta \\ \sin\theta & \cos\theta & -x\sin\theta+y\cos\theta \\ x & -y & 1 \end{bmatrix} \begin{bmatrix} g_{XX} & g_{XY} & g_{XZ} \\ g_{YX} & g_{YY} & g_{YZ} \\ g_{ZX} & g_{ZY} & g_{ZZ} \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & \sin\theta & x \\ -\sin\theta & \cos\theta & -y \\ -x\cos\theta-y\sin\theta & -x\sin\theta+y\cos\theta & 1 \end{bmatrix}$$

where the index over the gravity gradients again indicates the reference frame in which they are coordinatized. The manipulation of the above matrix equation gives for the six gravity gradients of interest the following:

$$\overset{I}{g}_{XX}=g_{XX}\cos^2\theta+g_{YY}\sin^2\theta+g_{ZZ}(x^2\cos^2\theta+y^2\sin^2\theta)-g_{XY}\sin2\theta-2g_{XZ}x\cos^2\theta-g_{XZ}y\sin2\theta+g_{ZY}x\sin2\theta+$$
$$+g_{ZZ}xy\sin\theta+2g_{ZY}y\sin^2\theta$$

$$\overset{I}{g}_{XY}=g_{XX}\cos\theta\sin\theta+g_{XY}(1-2\sin^2\theta)+g_{XZ}(y-2y\sin^2\theta-x\cos\theta\sin\theta\ -x\cos^2\theta)-g_{YY}\sin\theta\cos\theta+$$
$$+g_{YZ}(x\sin\theta\cos\theta-y\sin2\theta-x\cos^2\theta)+g_{ZZ}(x^2\cos^2\theta-xy\cos^2\theta+xy\sin\theta\cos\theta-y^2\sin\theta\cos\theta)$$

$$\overset{I}{g}_{XZ}=g_{XX}x\cos\theta-g_{XY}(x\sin\theta+y\cos\theta)-g_{XZ}(x^2\cos\theta-\cos\theta+xy\sin\theta)+g_{YY}y\sin\theta-g_{ZZ}(x\cos\theta+\sin\theta)+$$
$$\text{(7.2.23a)}\qquad +g_{ZY}(xy\cos\theta+xy\sin\theta-\sin\theta)$$

$$\overset{I}{g}_{YY}=g_{XX}\sin^2\theta+g_{XY}\sin2\theta+g_{XZ}(y\sin2\theta-2x\sin^2\theta)+g_{YY}(2y\cos^2\theta-x\sin\theta)+g_{ZZ}(x^2\sin^2\theta+y^2\cos^2\theta-$$
$$-xy\sin\theta)$$

$$\overset{I}{g}_{YZ}=g_{XX}x\sin\theta+g_{XY}(x\cos\theta-y\sin\theta)+g_{XZ}(\sin\theta-x^2\sin\theta+xy\cos\theta)-g_{YY}y\cos\theta+g_{YZ}(xy\sin\theta-y^2\cos\theta+\cos\theta)+$$
$$+g_{ZZ}(y\cos\theta-x\sin\theta)$$

$$\overset{I}{g}_{ZZ}=g_{XX}x^2-2g_{XY}xy+2g_{XZ}x+g_{YY}y^2-2g_{YZ}y+g_{ZZ}$$

where all gravity gradients in the right hand side refer to the navigation frame. Eqs(7.2.23a) written in matrix form read(see next page).

As we have already discussed, the first procedure to be followed in trying to simulate the fundamental equation of inertial navigation is to express all vector quantities to the same coordinate frame and particularly to the navigation frame in order the results to be referred to the earth-linked frame. As far as gravity gradients are concerned, the following transformations must be made to "channel" the signal as it is sensed by the electronic float axes to that referred to the navigation frame:

1)rewrite eqs(7.2.18), (7.2.22b) and (7.2.23b) absractly as

$$\text{(7.2.18)}'\quad \overset{E}{g}_{ij} = [A] + [B]\,\overset{G}{g}_{ij}$$

$$\text{(7.2.22b)}'\quad \overset{I}{g}_{ij} = [C]\,\overset{G}{g}_{ij}$$

$$\text{(7.2.23b)}'\quad \overset{I}{g}_{ij} = [D]\,\overset{N}{g}_{ij}$$

$$\begin{bmatrix} g^{I}_{XX} \\ g^{I}_{XY} \\ g^{I}_{XZ} \\ g^{I}_{YY} \\ g^{I}_{YZ} \\ g^{I}_{ZZ} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & -\sin 2\theta & -2x\cos^2\theta - y\sin 2\theta & \sin^2\theta & x\sin 2\theta + 2y\sin^2\theta & x^2\cos^2\theta + y^2\sin^2\theta + xy\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \cos^2\theta - \sin^2\theta & y - 2y\sin^2\theta - x\cos\theta\sin\theta - x\cos^2\theta & -\frac{1}{2}\sin 2\theta & x\sin\theta\cos\theta - y\sin\theta - x\cos^2\theta & x^2\cos^2\theta - xy\cos^2\theta + \frac{1}{2}\sin 2\theta(xy - y^2) \\ x\cos\theta & x\sin\theta - y\cos\theta & -x^2\cos\theta + \cos\theta - xy\sin\theta & y\sin\theta & xy\cos\theta - \sin\theta + xy\sin\theta & -x\cos\theta - y\sin\theta \\ \sin^2\theta & \sin 2\theta & y\sin\theta - 2x\sin^2\theta & \cos^2\theta & 2y\cos^2\theta - x\sin 2\theta & x^2\sin^2\theta + y^2\cos^2\theta - xy\sin\theta \\ x\sin\theta & x\cos\theta - y\sin\theta & \sin\theta - x^2\sin\theta + xy\cos\theta & -y\cos\theta & xy\sin\theta + \cos\theta - y^2\cos\theta & y\cos\theta - x\sin\theta \\ x^2 & -2xy & 2x & y^2 & -2y & 1 \end{bmatrix} \begin{bmatrix} g^{N}_{XX} \\ g^{N}_{XY} \\ g^{N}_{XZ} \\ g^{N}_{YY} \\ g^{N}_{YZ} \\ g^{N}_{ZZ} \end{bmatrix} \quad (7.2.23b)$$

where the g's represent the column matrices of the gravity gradients and A,B,C,D the already defined transformation matrices.

2)derive those gravity gradients referred to the navigation frame using the following formula:

$$(7.2.24)\quad \overset{N}{g}_{ij} = [D]^{-1}\,[C]([B]^{-1}(\overset{E}{g}_{ij} - [A]))$$

where the superscript -1 indicates the inverse matrix operation. The last equation, in view of the involved matrices, shows clearly that the sensed gravity gradients should undergo a very tedious and lengthy manipulation containing non-orthogonality effects, instrumental misalignment etc. in order to be finally used in the simulated navigation equation to be presented next.

7.3 The simulated navigation equation

In view of eqs(2.11),(7.1.12),(7.1.13) and (7.2.24) the complete navigation equation which can estimate the instantaneous geocentric coordinates of the moving object with respect to the earth-linked navigation frame, is written in matrix form:

$$(7.3.1)\quad \ddot{R}_E = T(aA^{in}+b+u)+D^{-1}C(B^{-1}(g^E_{ij}-A))-2\Omega^I_E\dot{R}_E-\dot{\Omega}^I_E xR-\Omega^I_E x(\Omega^I_E xR)$$

where the above symbology has been already defined. Now, in order to get out of the navigation system some indicated numbers or order of magnitude of errors, certain assumptions have to be made(of course, a rigorous statistical analysis should include the full matrices involved):

a) Since the navigation system is simulated for its performance during a very limited time span, or as a matter of fact for a few seconds, it is reasonable to neglect all terms containing angular velocity or acceleration of the earth coordinatized in the inertial space. Consequently, the last three terms in eq.(7.3.1) are for our simulation studies dropped out.

b) The matrix T given in eq.(7.1.12) contains products of small angles plus some single terms. Having decided to keep only first order terms, then the matrix T can be approximated by:

$$(7.3.2) \qquad T = \begin{bmatrix} 1 & -(\varepsilon_{XZ}+m_Z) & \varepsilon_{XZ}+m_Y \\ m_Z+\varepsilon_{YZ} & 1 & -(\varepsilon_{YX}+m_X) \\ -(m_Y+\varepsilon_{ZY}) & m_X+\varepsilon_{ZX} & 1 \end{bmatrix} + \text{higher order terms}$$

Combining eqs(7.1.13) and (7.1.2), we get for the apparent acceleration components to be used inside the navigation equation the following:

$$(7.3.3) \qquad \begin{bmatrix} A_X^a \\ A_Y^a \\ A_Z^a \end{bmatrix} = \begin{bmatrix} (1+a_X)A_X^{in}+b_X+u_X-(\varepsilon_{XZ}+m_Z)A_Y^{in}+(\varepsilon_{XZ}+m_Y)A_Z^{in} \\ (m_Z+\varepsilon_{YZ})A_X^{in}+(1+a_Y)A_Y^{in}+b_Y+u_Y-(\varepsilon_{YX}+m_Z)A_Z^{in} \\ -(m_Y+\varepsilon_{ZY})A_X^{in}+(m_X+\varepsilon_{ZX})A_Y^{in}+(1+a\ )A_Z^{in}+b_Z+u_Z \end{bmatrix} + \text{higher order terms}$$

The above equation shows clearly that the components of the apparent acceleration in each channel are a mixture of all three encountered acceleration components.As it can be also seen, each channel's signal is merely composed of its counterpart acceleration and the other two signals multiplied by small quantities(due to the accelerometer non-orthogonality) are present as well.

c) Let us now consider the case of the gravity gradients. As eq.(7.2) shows the determination of the gravity gradients to be used in the navigation equation requires the inversion of two matrices, namely the D^{-1} and B^{-1}. Since we have first to derive the final error matrix in front of the gravity gradients referred to the electronic frame and then to make the assumptions, the mentioned inversions must be computed by hand. B^{-1} presents no difficulties, but D^{-1} possesses certain problems due to its elements' complexity. Owing to this fact, eq.(7.2.22b)' is considered as

$$g^G = C^{-1}\, g^I$$

Therefore

$$g^E = A + BC^{-1} g^I = A + BC^{-1} D g^N = \Omega g^N$$

and finally

(7.3.4) $g^N = \Omega^{-1} g^E$

The inversion of C^{-1} by the partitioning method gives the following matrix approximated only up to the first order terms:

$$(7.3.5)\quad C^{-1} = \begin{bmatrix} 1 & -2\xi_Z & 2\xi_Y & 0 & 0 & 0 \\ \xi_Z & 1 & \xi_X & -\xi_Z & \xi_Y & 0 \\ -\xi_Y & \xi_X & 1 & 0 & -\xi_Z & \xi_Y \\ 0 & 2\xi_Z & 0 & 1 & -2\xi_X & 0 \\ 0 & -\xi_Y & \xi_Z & \xi_X & 1 & -\xi_X \\ 0 & 0 & -2\xi_Y & 0 & 2\xi_X & 1 \end{bmatrix} + \text{higher order terms}$$

Therefore, the transformation(7.3.4) yields the following:

$$g^N_{XX} = g^E_{XX} + g^E_{XY}\cdot 2\theta - \left(\frac{8}{3}\pi k\rho - \frac{4}{3}\omega^2\right)$$

$$g^N_{XY} = g^E_{XY} + g^E_{XX}\cdot\theta + g^E_{XZ}\cdot y + g^E_{YY}\cdot\theta + \frac{1}{2}(\theta_{XZ}+\theta_{YZ})g^E_{ZZ} - \left(\frac{8}{3}\pi k\rho - \frac{4}{3}\omega^2\right)$$

$$(7.3.6)\quad g^N_{XZ} = g^E_{XZ} + g^E_{YZ}\cdot\theta + \left(\psi_2 - \xi_Y + \frac{1}{2}(\theta_{YZ}-\theta_{XZ})\right)g^E_{ZZ}$$

$$g^N_{YY} = g^E_{YY} - g^E_{XY}\cdot 2\theta - \left(\frac{8}{3}\pi k\rho - \frac{4}{3}\omega^2\right)$$

$$g^N_{YZ} = g^E_{YZ} + \left(\xi_X - \psi_1 + \frac{1}{2}(\theta_{XZ}-\theta_{YZ})\right)g^E_{ZZ}$$

$$g^N_{ZZ} = g^E_{ZZ} - g^E_{XZ}\cdot 2x + g^E_{YZ}\cdot 2y - \left(\frac{8}{3}\pi k\rho - \frac{4}{3}\omega^2\right)$$

As we can see from the above expressions, the gravity gradients referred to the navigation frame are equal to their counterparts sensed and measured by the electronic frame plus products of gravity gradients with small quantities e.g. polar motion components. All these terms in view of 1)the conclusions drawn in simulation I with regard to gravity gradients 2)the small magnitude of all terms but the single gravity gradients and 3)the objective of our simulation studies, are finally neglected. Consequently, the gravity gradients which enter the simulated navigation equation are those which are measured by the electronic float frame.

Recollecting results, the equations which we shall simulate assume the

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following forms:

$$
\begin{aligned}
X_3 &= \Delta t^2((1+a_X)A^{in}_{X_1}+b_X+u_X-(\epsilon_{XZ}+m_Z)A^{in}_{Y_1}+(\epsilon_{XZ}+m_Y)A^{in}_{Z_1}+G_{X_1}+G_{XX_1}(X_2-X_1)+G_{XY_1}(Y_2-Y_1)+\\
&\quad +G_{XZ_1}(Z_2-Z_1))+2X_2-X_1\\
Y_3 &= \Delta t^2((m_Z+\epsilon_{YZ})A^{in}_{X_1}+(1+a_Y)A^{in}_{Y_1}+b_Y+u_Y-(\epsilon_{YX}+m_X)A^{in}_{Z_1}+G_{Y_1}+G_{YX_1}(X_2-X_1)+G_{YY_1}(Y_2-Y_1)+\\
&\quad +G_{YZ_1}(Z_2-Z_1))+2Y_2-Y_1\\
Z_3 &= \Delta t^2(-(m_Y+\epsilon_{ZY})A^{in}_{X_1}+(m_X+\epsilon_{ZX})A^{in}_{Y_1}+(1+a_Z)A^{in}_{Z_1}+b_Z+u_Z+G_{Z_1}+G_{ZX_1}(X_2-X_1)+G_{ZY_1}(Y_2-Y_1)+\\
&\quad +G_{ZZ_1}(Z_2-Z_1))+2Z_2-Z_1
\end{aligned}
\tag{7.3.7}
$$

The above equations contain 36 parameters, namely:

a)time span(1)
b)apparent acceleration components(3)
c)the geocentric coordinates of the first two points(6)
d)the gravity components(3)
e)the gravity gradients(6)
f)the accelerometer bias(3)
g)the accelerometer random uncertainty(3)
f)the accelerometer non-orthogonality(5)
i)the initial misalignment angles(3)
j)the accelerometer scale factor uncertainty(3)

The same procedure followed in simulation I will be repeated to see the influence of these 36 parameters on the derived coordinates

7.3 Results

The same computer programme used for simulation I studies is employed to accommodate the new simulated equations. Only the results for the parameters which can contribute changes into the system's error budget are listed below.

$\sigma^2_{X_1}$ = (m²)	var.-cov.: (m²)		
0.00	4.0029	0.0029	0.0437
		5.0029	0.0437
			5.6626
0.01	4.0129	0.0029	0.0437
		5.0029	0.0437
			5.6626
0.05	4.0529	0.0029	0.0437
		5.0029	0.0437
			5.6626
0.10	4.1029	0.0029	0.0437
		5.0029	0.0437
			5.6626
0.20	4.2029	0.0029	0.0437
		5.0029	0.0437
			5.6626
0.50	4.5029	0.0029	0.0437
		5.0029	0.0437
			5.6626
2.00	6.0029	0.0029	0.0437
		5.0029	0.0437
			5.6626
5.00	9.0029	0.0029	0.0437
		5.0029	0.0437
			5.6626

$\sigma^2_{X_2}$ = (m²)	var.-cov.: (m²)		
0.00	1.0000	0.0000	0.0002
		5.0001	0.0002
			5.0034
0.01	1.0400	0.0000	0.0002
		5.0001	0.0002
			5.0034
0.05	1.2000	0.0000	0.0002
		5.0001	0.0002
			5.0034
0.10	1.4000	0.0000	0.0002
		5.0001	0.0002
			5.0034
0.20	1.8000	0.0000	0.0002
		5.0001	0.0002
			5.0034
0.50	3.0000	0.0000	0.0002
		5.0001	0.0002
			5.0034
2.00	9.0000	0.0000	0.0002
		5.0001	0.0002
			5.0034
5.00	21.0000	0.0000	0.0002
		5.0001	0.0002
			5.0034

$\sigma^2_{Z_1}$ = (m^2)	var.-cov.: (m^2)		
	5.0000	0.0000	0.0002
0.00		5.0001	0.0002
			4.0034
	5.0000	0.0000	0.0002
0.01		5.0001	0.0002
			4.0134
	5.0000	0.0000	0.0002
0.05		5.0001	0.0002
			4.0234
	5.0000	0.0000	0.0002
0.10		5.0001	0.0002
			4.0534
	5.0000	0.0000	0.0002
0.20		5.0001	0.0002
			4.1034
	5.0000	0.0000	0.0002
0.50		5.0001	0.0002
			4.5034
	5.0000	0.0000	0.0002
2.00		5.0001	0.0002
			6.0034
	5.0000	0.0000	0.0002
5.00		5.0001	0.0002
			9.0034

$\sigma^2_{Z_2}$ = (m^2)	var.-cov.: (m^2)		
	5.0000	0.0000	0.0002
0.00		5.0001	0.0002
			1.0034
	5.0000	0.0000	0.0002
0.01		5.0001	0.0002
			1.0434
	5.0000	0.0000	0.0002
0.05		5.0001	0.0002
			1.2034
	5.0000	0.0000	0.0002
0.10		5.0001	0.0002
			1.4034
	5.0000	0.0000	0.0002
0.20		5.0001	0.0002
			1.8034
	5.0000	0.0000	0.0002
0.50		5.0001	0.0002
			3.0034
	5.0000	0.0000	0.0002
2.00		5.0001	0.0002
			9.0034
	5.0000	0.0000	0.0002
5.00		5.0001	0.0002
			21.0034

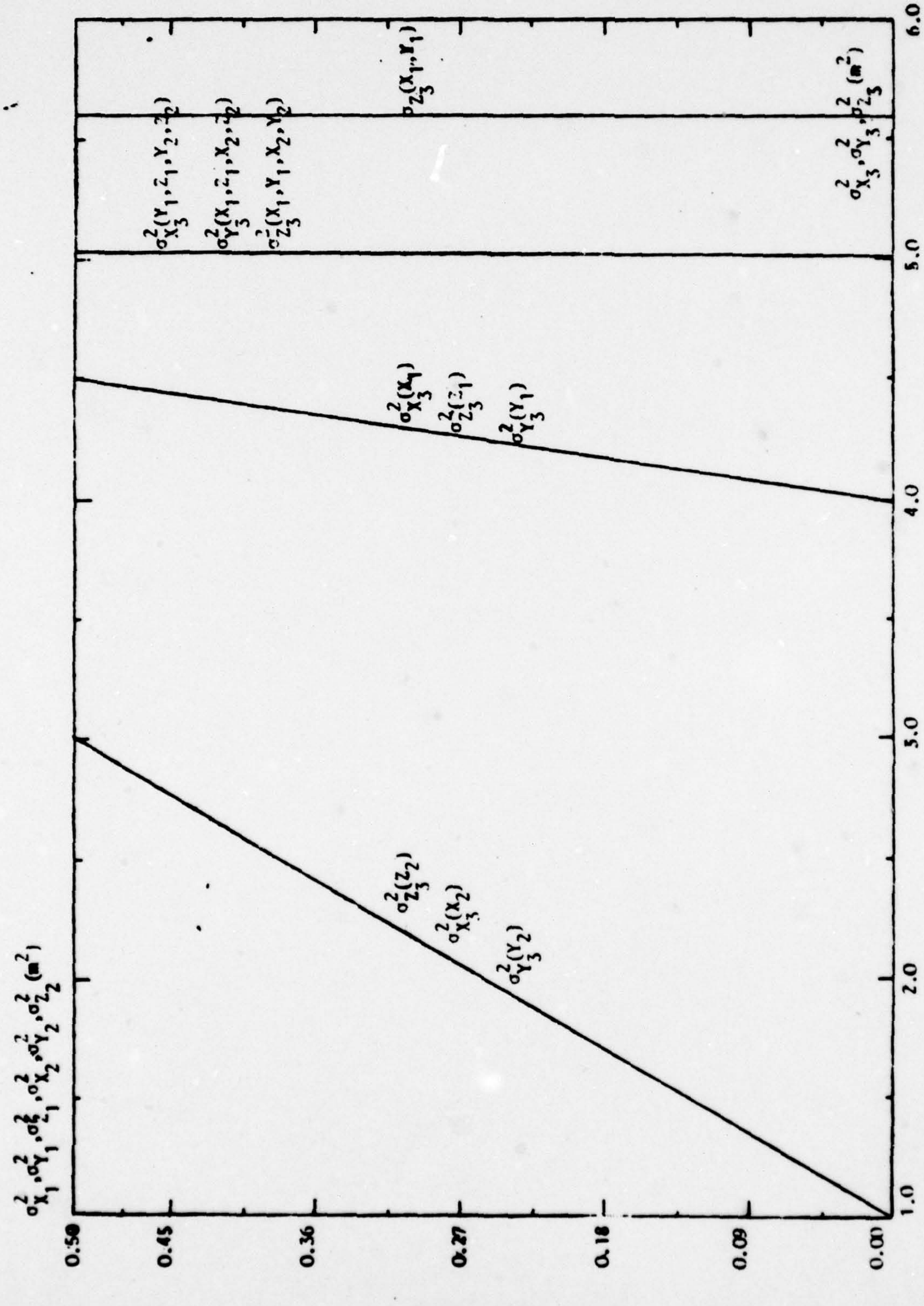
$\sigma^2_{A_X}$ = (m^2/s^4)	var.-cov.:		(m^2)
	5.0000	0.0000	0.0002
0.02-0.50		5.0001	0.0002
			5.0034

$\sigma^2_{A_Z}$ = (m^2/s^4)	var.-cov.:		(m^2)
	5.0000	0.0000	0.0002
0.02-0.50		5.0001	0.0002
			5.0034

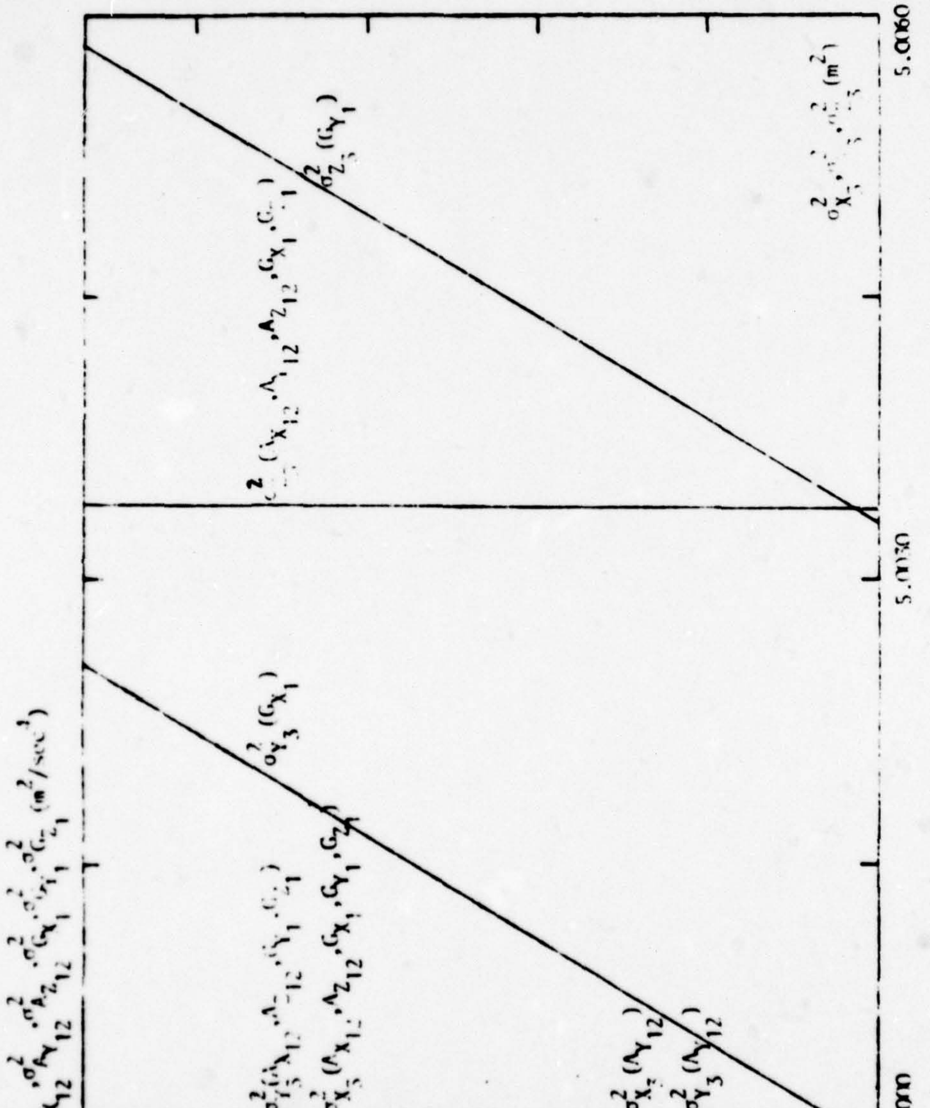
$\sigma^2_{G_X}$ = (m^2/s^4)	var.-cov.:		(m^2)
	5.0000	0.0000	0.0002
0.005		5.0000	0.0002
			5.0034
	5.0000	0.0001	0.0002
0.02		5.0001	0.0002
			5.0034
	5.0000	0.0002	0.0002
0.05		5.0003	0.0002
			5.0034
	5.0000	0.0003	0.0002
0.10		5.0006	0.0002
			5.0034
	5.0000	0.0006	0.0002
0.20		5.0012	0.0002
			5.0034
	5.0000	0.0009	0.0002
0.30		5.0018	0.0002
			5.0034
	5.0000	0.0012	0.0002
0.40		5.0024	0.0002
			5.0034

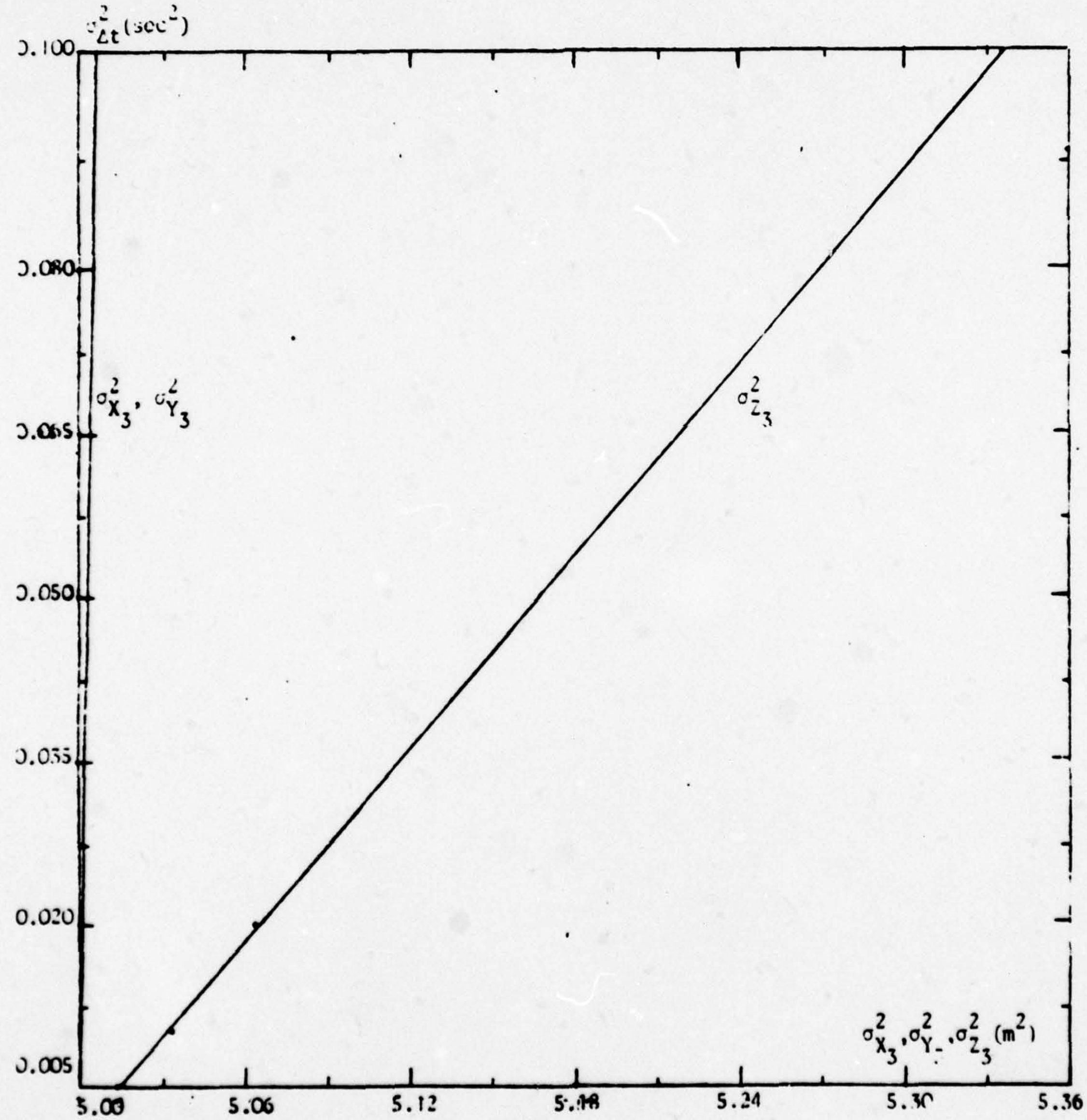
$\sigma^2_{G_Y}$ = (m^2/s^4)	var.-cov.:		(m^2)
	5.0000	0.0000	0.0002
0.005		5.0001	0.0002
			5.0033
	5.0000	0.0000	0.0003
0.02		5.0001	0.0002
			5.0033
	5.0000	0.0000	0.0004
0.05		5.0001	0.0002
			5.0036
	5.0000	0.0000	0.0005
0.10		5.0001	0.0002
			5.0039
	5.0000	0.0000	0.0008
0.20		5.0001	0.0002
			5.0045
	5.0000	0.0000	0.0011
0.30		5.0001	0.0002
			5.0051
	5.0000	0.0000	0.0014
0.40		5.0001	0.0002
			5.0057

$\sigma^2_{\Delta t}=$ (s^2)	var.-cov.: (m^2)		
	5.0000	0.0000	0.0001
0.0001		5.0001	0.0000
			5.0004
	5.0000	0.0000	0.0001
0.0005		5.0001	0.0001
			5.0017
	5.0001	0.0001	0.0011
0.005		5.0001	0.0011
			5.0166
	5.0001	0.0002	0.0022
0.01		5.0002	0.0022
			5.0332
	5.0003	0.0003	0.0044
0.02		5.0003	0.0044
			5.0663
	5.0007	0.0008	0.0110
0.05		5.0008	0.0109
			5.1657
	5.0014	0.0015	0.0219
0.1		5.0015	0.0218
			5.3313
	5.0029	0.0029	0.0437
0.2		5.0029	0.0437
			5.6626



$\sigma^2_{X_1}, \sigma^2_{Y_1}, \sigma^2_{Z_1}, \sigma^2_{X_2}, \sigma^2_{Y_2}, \sigma^2_{Z_2}$ (m^2)
0.50
0.45
0.36
0.27
0.18
0.09
0.00
1.0
2.0
5.0
4.0
5.0
6.0
$\sigma^2_{Z_3}(Z_2)$
$\sigma^2_{X_3}(X_2)$
$\sigma^2_{Y_3}(Y_2)$
$\sigma^2_{X_3}(X_1)$
$\sigma^2_{Z_3}(Z_1)$
$\sigma^2_{Y_3}(Y_1)$
$\sigma^2_{X_3}(Y_1, Z_1, Y_2, Z_2)$
$\sigma^2_{Y_3}(X_1, Z_1, X_2, Z_2)$
$\sigma^2_{Z_3}(X_1, Y_1, X_2, Y_2)$
$\sigma_{Z_3}(X_1, Y_1)$
$\sigma^2_{X_3}, \sigma^2_{Y_3}, \sigma^2_{Z_3}$ (m^2)


0.42
0.36
0.27
0.18
0.09
0.00
5.0000
5.0030
5.0060


σ²Δt (sec²)
0.100
0.080
0.065
0.050
0.035
0.020
0.005
σ²X3, σ²Y3
σ²Z3
σ²X3, σ²Y3, σ²Z3 (m²)
5.00
5.06
5.12
5.18
5.24
5.30
5.36

7.4 Discussion

Needless to note that the results of the new simulation are almost identical to those derived in simulation I and therefore the same comments are also applicable here. But since we are not at all satisfied with the system's behaviour, we try to investigate the aided navigation system deeper by making the following studies:

a) Investigation on the contribution of the sampling interval Δt: to motivate our discussion we remind again that the inertial acceleration components have been approximated by the Stirling's formula (see eq. (5.2.3)). In approximating derivatives a dominant error source is the input errors themselves. One could immediately see the explanation to that looking at the mentioned equation. The reciprocal power of the sampling interval Δt multiplies the true values as well as their errors and thus the algorithm magnifies them enormously. For that reason we investigate the case in which Δt decreases in order to see how space traverses of such an aided navigation system behave. The results are listed in the next page.

From the given results it is obvious that:

1) The navigation system becomes a little more tolerable as Δt decreases but again it behaves badly

2) If we call point 10 of the first traverse as the "break point" of the navigation system, then it occurs at point 11 and 13 for the rest traverses.

Consequently, by making Δt very small the system does not behave better except some very small improvements.

b) Investigation on the contribution of the omitted covariances: from the given results of simulation studies I and II it is evident that the system is very sensitive to the variances of the initial coordinates and especially to the variances with which the geocentric coordinates of the second point are known. Therefore, we try to see if the omission of the respective covariances makes any changes into the system's error budget. The formulation of the new investigation follows these brief lines:

1) rewrite the general simulated equations as:

Point	Δt=0.1sec.	Δt=0.05sec.	Δt=0.01sec.
		σ_X^2 : σ_Y^2 : σ_Z^2 : (m^2)	
3	$0.1549\ 10^{-4}$ $0.7540\ 10^{-4}$ $0.3373\ 10^{-2}$	$0.3365\ 10^{-5}$ $0.5412\ 10^{-5}$ $0.8299\ 10^{-3}$	$0.1441\ 10^{-6}$ $0.1446\ 10^{-6}$ $0.3313\ 10^{-4}$
4	$0.7745\ 10^{-4}$ $0.3370\ 10^{-3}$ $0.1687\ 10^{-1}$	$0.1715\ 10^{-4}$ $0.2706\ 10^{-4}$ $0.4150\ 10^{-2}$	$0.7041\ 10^{-6}$ $0.7230\ 10^{-6}$ $0.1656\ 10^{-3}$
5	$0.3408\ 10^{-3}$ $0.1659\ 10^{-2}$ $0.7423\ 10^{-1}$	$0.7564\ 10^{-4}$ $0.1195\ 10^{-3}$ $0.1826\ 10^{-1}$	$0.3084\ 10^{-5}$ $0.3181\ 10^{-5}$ $0.7287\ 10^{-3}$
6	$0.1456\ 10^{-2}$ $0.7089\ 10^{-2}$ 0.3172	$0.3234\ 10^{-3}$ $0.5106\ 10^{-3}$ $0.7802\ 10^{-1}$	$0.1316\ 10^{-4}$ $0.1359\ 10^{-4}$ $0.3114\ 10^{-2}$
7	$0.6180\ 10^{-2}$ $0.3009\ 10^{-1}$ 1.3460	$0.1373\ 10^{-2}$ $0.2167\ 10^{-2}$ 0.3312	$0.5586\ 10^{-4}$ $0.5769\ 10^{-4}$ $0.1322\ 10^{-1}$
8	$0.2619\ 10^{-1}$ 0.1275 5.7050	$0.5819\ 10^{-2}$ $0.9184\ 10^{-2}$ 1.4041	$0.2367\ 10^{-3}$ $0.2445\ 10^{-3}$ $0.5603\ 10^{-1}$
9	0.1110 0.5402 24.1707	$0.2465\ 10^{-1}$ $0.3891\ 10^{-1}$ 5.9481	$0.1003\ 10^{-2}$ $0.1036\ 10^{-2}$ 0.2374
10	0.4702 2.2886 102.4000	0.1044 0.1648 25.2002	$0.4249\ 10^{-2}$ $0.4389\ 10^{-2}$ 1.0060
11		0.4423 0.6918 106.7000	$0.1800\ 10^{-1}$ $0.1859\ 10^{-1}$ 4.2610
12			$0.7623\ 10^{-1}$ $0.7875\ 10^{-1}$ 18.0500

$$X_i = \Delta t^2 \left(A_{X_{i-2,i-1}} + G_{X_{i-2}} + G_{XX_{i-2,i-1}} (X_{i-1}-X_{i-2}) + G_{XY_{i-2,i-1}} (Y_{i-1}-Y_{i-2}) + \right.$$

$$\left. G_{XZ_{i-2,i-1}} (Z_{i-1}-Z_{i-2}) \right) + 2X_{i-2} - X_{i-1}$$

$$Y_i = \Delta t^2 \left(A_{Y_{i-2,i-1}} + G_{Y_{i-2}} + G_{YX_{i-2,i-1}} (X_{i-1}-X_{i-2}) + G_{YY_{i-2,i-1}} (Y_{i-1}-Y_{i-2}) + \right.$$

$$\left. G_{YZ_{i-2,i-1}} (Z_{i-1}-Z_{i-2}) \right) + 2Y_{i-2} - Y_{i-1}$$

$$Z_i = \Delta t^2 \left(A_{Z_{i-2,i-1}} + G_{Z_{i-2}} + G_{ZX_{i-2,i-1}} (X_{i-1}-X_{i-2}) + G_{ZY_{i-2,i-1}} (Y_{i-1}-Y_{i-2}) + \right.$$

$$\left. G_{ZZ_{i-2,i-1}} (Z_{i-1}-Z_{i-2}) \right) + 2Z_{i-2} - Z_{i-1}$$

2)call y the parameters of the above equation except the coordinates and C the desired coordinates. Then for the first point one gets:

$$\begin{bmatrix} X_3 & Y_3 & Z_3 \end{bmatrix}^T = \begin{bmatrix} A_3 \end{bmatrix} \begin{bmatrix} y \end{bmatrix}$$

$$\Sigma_{C_3} = A_3 \Sigma_y A_3^T$$

since $\mathrm{var}(X_1, Y_1, Z_1, X_2, Y_2, Z_2) = 0$. Σ_{C_3} indicates the dispersion matrix of the coordinates at point 3.

<u>point 4</u>

$$\begin{bmatrix} C_4 \end{bmatrix} = A_4 y_4 + B_4 C_3 = \begin{bmatrix} A_4 & B_4 \end{bmatrix} \Sigma \begin{bmatrix} y_4 \\ C_3 \end{bmatrix}$$

$$\Sigma_{C_4} = \begin{bmatrix} A_4 & B_4 \end{bmatrix} \Sigma_{\begin{bmatrix} y_4 \\ C_3 \end{bmatrix}} \begin{bmatrix} A_4 & B_4 \end{bmatrix}^T \qquad \Sigma_{\begin{bmatrix} y_4 \\ C_3 \end{bmatrix}} = \begin{bmatrix} \Sigma_{y_4} & 0 \\ 0 & \Sigma_{C_3} \end{bmatrix}$$

$$\Sigma_{C_4} = \begin{bmatrix} A_4 & B_4 \end{bmatrix} \begin{bmatrix} \Sigma_{y_4} & 0 \\ 0 & \Sigma_{C_3} \end{bmatrix} \begin{bmatrix} A_4 & B_4 \end{bmatrix}^T$$

<u>point 5</u>

$$\begin{bmatrix} C_5 \end{bmatrix} = A_5 y_5 + B_5 C_4 + D_5 C_3 = \begin{bmatrix} A_5 & B_5 & D_5 \end{bmatrix} \begin{bmatrix} y_5 \\ C_4 \\ C_3 \end{bmatrix}$$

$$\Sigma_{C_5} = \begin{bmatrix} A_5 & B_5 & D_5 \end{bmatrix} \Sigma_{\begin{bmatrix} y_5 \\ C_4 \\ C_3 \end{bmatrix}} \begin{bmatrix} A_5 & B_5 & D_5 \end{bmatrix}^T$$

$$\Sigma_{\begin{bmatrix} y_5 \\ C_4 \\ C_3 \end{bmatrix}} = \begin{bmatrix} \Sigma_{y_5} & 0 \\ 0 & \Sigma_{\begin{bmatrix} C_4 \\ C_3 \end{bmatrix}} \end{bmatrix} \qquad \begin{bmatrix} C_4 \\ C_3 \end{bmatrix} \begin{bmatrix} A_4 & B_4 & y_4 \\ 0 & I & C_3 \end{bmatrix}$$

$$\Sigma_{\begin{bmatrix} C_4 \\ C_3 \end{bmatrix}} = \begin{bmatrix} A_4 & B_4 \\ 0 & I \end{bmatrix} \Sigma_{\begin{bmatrix} y_4 \\ C_3 \end{bmatrix}} \begin{bmatrix} A_4 & B_4 \\ 0 & I \end{bmatrix}^T$$

$$\Sigma_{C_5} = \begin{bmatrix} A_5 & B_5 & D_5 \end{bmatrix} \begin{bmatrix} \Sigma_{y_5} & 0 \\ 0 & \Sigma_{\begin{matrix} C_4 \\ C_3 \end{matrix}} \end{bmatrix} \begin{bmatrix} A_5 & B_5 & D_5 \end{bmatrix}^T$$

It can be seen that from point 5 the first covariances between the coordinates are involved.

<u>point 6</u>

$$\begin{bmatrix} C_6 \end{bmatrix} = A_6 y_6 + B_6 C_5 + D_6 C_4 = \begin{bmatrix} A_6 & B_6 & D_6 \end{bmatrix} \begin{bmatrix} y_6 \\ C_5 \\ C_4 \end{bmatrix}$$

$$\Sigma_{C_6} = \begin{bmatrix} A_6 & B_6 & D_6 \end{bmatrix} \Sigma_{\begin{bmatrix} y_6 \\ C_5 \\ C_4 \end{bmatrix}} \begin{bmatrix} A_6 & B_6 & D_6 \end{bmatrix}^T$$

$$\begin{bmatrix} C_5 \\ C_4 \end{bmatrix} = \begin{bmatrix} A_5 & B_5 & D_5 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} y_5 \\ C_4 \\ C_3 \end{bmatrix} \begin{bmatrix} A_5 & B_5 & D_5 \\ 0 & I & 0 \end{bmatrix}^T$$

and so on.

According to the above equations, a new gradiometer-aided navigation system space traverse is carried out. The results are listed below.

Point	var.-cov.: (m^2)		
3	$0.1638\ 10^{-4}$	$0.1438\ 10^{-4}$	$0.2182\ 10^{-3}$
		$0.1638\ 10^{-4}$	$0.2182\ 10^{-3}$
			$0.3314\ 10^{-2}$
4	$0.8038\ 10^{-4}$	$0.7190\ 10^{-4}$	$0.1091\ 10^{-2}$
		$0.8038\ 10^{-4}$	$0.1091\ 10^{-2}$
			$0.1657\ 10^{-1}$
5	$0.3543\ 10^{-3}$	$0.3164\ 10^{-3}$	$0.4599\ 10^{-2}$
		$0.3543\ 10^{-3}$	$0.4601\ 10^{-2}$
			$0.7345\ 10^{-1}$
6	$0.1514\ 10^{-2}$	$0.1352\ 10^{-2}$	$0.1971\ 10^{-1}$
		$0.1514\ 10^{-2}$	$0.2051\ 10^{-1}$
			0.3137
7	$0.6426\ 10^{-2}$	$0.5737\ 10^{-2}$	$0.8364\ 10^{-1}$
		$0.6426\ 10^{-2}$	$0.8708\ 10^{-1}$
			1.3315

From the given results it is evident that the omitted covariances do not play any critical role on the derived coordinate variances.

c) Investigation on a multipoint approximation: the last hope to improve the bad behaviour of the gradiometer-aided inertial navigation system is to include more terms for the Stirling's approximation formula and thus to approximate better the inertial acceleration components. Eq.(5.2.3) includes only the first term of the mentioned formula, but manipulation of the second term gives to the fundamental equation of inertial navigation to be simulated the following form:

$$X_i = 2\Delta t^2 (A_{X_{i-2,i-1}} + G_{X_{i-2}} + G_{XX_{i-2,i-1}} (X_{i-1} - X_{i-2}) + G_{XY_{i-2,i-1}} (Y_{i-1} - Y_{i-2}) + G_{XZ_{i-2,i-1}} (Z_{i-1} - Z_{i-2})) + 4X_{i-2} - 4X_{i-3} + X_{i-4} .$$

$$Y_i = 2\Delta t^2 (A_{Y_{i-2,i-1}} + G_{Y_{i-2}} + G_{XY_{i-2,i-1}} (X_{i-1} - X_{i-2}) + G_{YY_{i-2,i-1}} (Y_{i-1} - Y_{i-2}) + \quad (7.3.8)$$

$$+ G_{YZ_{i-2,i-1}} (Z_{i-1} - Z_{i-2})) + 4Y_{i-2} - 4Y_{i-3} + Y_{i-4}$$

$$Z_i = 2\Delta t^2 (A_{Z_{i-2,i-1}} + G_{Z_{i-2}} + G_{ZX_{i-2,i-1}} (X_{i-1} - X_{i-2}) + G_{ZY_{i-2,i-1}} (Y_{i-1} - Y_{i-2}) +$$

$$+ G_{ZZ_{i-2,i-1}} (Z_{i-1} - Z_{i-2})) + 4Z_{i-2} - 4Z_{i-3} + Z_{i-4}$$

The same procedure followed by the previous simulation studies is applied again to the above equations. The space traverse for the multipoint approximation gives the following results:

Point	var.-cov.: (m^2)		
5	$0.6560\ 10^{-4}$	$0.6560\ 10^{-4}$	$0.4177\ 10^{-3}$
6	$0.6560\ 10^{-4}$	$0.6560\ 10^{-4}$	$0.4177\ 10^{-3}$
7	$0.1152\ 10^{-3}$	$0.1152\ 10^{-3}$	$0.5186\ 10^{-3}$
8	$0.2165\ 10^{-3}$	$0.2165\ 10^{-3}$	$0.7100\ 10^{-2}$
9	$0.1961\ 10^{-2}$	$0.1961\ 10^{-2}$	$0.7105\ 10^{-2}$
10	$0.5320\ 10^{-2}$	$0.5320\ 10^{-2}$	$0.7518\ 10^{-2}$
11	$0.3497\ 10^{-1}$	$0.3497\ 10^{-1}$	$0.3218\ 10^{-1}$
12	0.1672	0.1672	0.1144
13	0.6466	0.6466	0.1273
14	2.4324	2.4324	0.1282
15	6.4771	6.4771	1.8289

From the above listed results it is evident that:

1) the navigation system's behaviour is now better
2) the tremendous instability of the Z-channel has been already diminished.
3) the system gathers less errors relative to the previous analysed traverses but it still needs to be updated at a certain navigation time after the initial observation point. This comes in confrontment with all up-to-date aided navigation systems which should be filtered out continuously as far as their committed errors are concerned. In the majority of inertial navigation applications, Kalman filtering is continuously applied (say, every some seconds of navigation time) to reduce the systems' inclination to gather errors which sometimes are intolerable.

8. Conclusions and Recommendations

We live undoubtedly in the space age. Everything moves quickly and accurately. Inertial navigation systems have been already introduced to accomplish the latter but, for the time being, in a limited sense. Well-known that systems which can guide a moving vehicle are burdened with errors(sometimes very big ones) and it is depending on the objective of the mission that the guiding system can be considered to be successful or not. For example, the strict requirements imposed on a satellite guidance system are not applicable in a slow moving vehicle such as a submarine. Thus, the system's efficiency will be depending on the objective of the guided object. Under that prism we analyse an aided navigation system in case of terrestrial navigation and especially when the moving vehicle is a cruise aircraft. The external aid is consisted of three mutually perpendicular gradiometers, a revoluntary equipment with the capability to measure the gravity gradient field of the space in which the navigation takes place. Several aids can be introduced instead of the gradiometers but we have chosen them in order to disperse possible fears about their usefulness of operation in the investigated application.

The inertially referenced acceleration of a moving vehicle can be obtained by adding the gravitational and non-gravitational (or apparent)acceleration and neglecting certain small terms which have been already discussed previously. Approximating the second inertial derivative with the Stirling's formula and then solving the resulting equations with respect to the unknown coordinates, the error propagation law can be applied through in order to get the expressions to be simulated. The analysis proves that a gradiometer-aided inertial navigation system is very unstable like all navigation systems. Particularly, the Z-channel gathers the biggest ammount of errors relative to the rest two ones and after some seconds the navigation system, as an instrumental package, collapses as far as its performance is concerned. Aiding the system with a barometer or an altimeter or generally with an instrument which can produce any kind of height information of the instantaneous position of the moving vehicle, then the instability of the discussed channel can be effectively reduced. Relevant to the two parameters of great interest, namely the acceleration and gravity ones, the former causes to the system an error up to the order of 7%(considering its contribution for some first points) and the latter a very small percentage of error. These results can justify the conclusion that gravity gradi-

ometers perform or rather behave excellent on-board a moving vehicle, such as an aircraft, or that the gravity-induced position error is a negligible small quantity.

Considering detailed error models for acceleration and gravity gradient measurements, a new statistical analysis proves that the aided navigation system under investigation remains unstable in almost the same fashion, accelerometer measurements errors are reduced down to 10% with respect to the previously derived numbers and gradiometers continue to fit excellent on-board. Trying to go out of this undesired "cul-de-sac" three new investigations are carried out:

a)Making the operation of the system as fine as possible, then the behaviour of the navigation system does not change appreciably. It will break down sooner or later.

b)In both simulation studies the initial value problem of inertial navigation is under consideration. The big influence of the two initial sets of coordinates on the derived accuracy of the system's output, justifies the hope that something could lie inside the omitted covariances between the coordinates of the two first points. The statistical analysis shows that the navigation system is again dying out in exactly the same manner as previously.

c)Since the whole analysis is based upon the approximation made, namely the Stirling's formula, it is reasonable to investigate the case in which more terms are taken into account. The simulated equations change format including now more terms(or more initial points). The new statistical analysis justifies the hopes that the system cannot be only badly-behaved but it possesses the ability to be more accurate or for that matter usable. The Z-channel instability is decreased effectively, the system gathers less errors than previously and it can now be seen with hope.

In spite of the not so accurate approximation for the second inertial derivatives, the quantization error to be committed is a very small quantity relative to the total error budget. Trying to give an order of magnitude for that error,I could simply mention that if the navigation system operates every 0.1sec., then after 20 minutes of flight the quantization error is up to the order of 4cm.

Further studies on that subject are advisable according to the following

general ideas:

1)Try to include more terms in approximating the second inertial derivatives in deriving the simulated equations for the first two or three unknown points. Simulate them and revise them for the next few points but include now more terms as far as the Stirling's formula is concerned. Apply this procedure continuously until the system has gathered errors which cannot be further tolerated. Then a filtering technique can"refresh" the system in order to begin again the above discussed procedure. In view of the obtained results of the multipoint approximation studies, I strongly believe that the proposed analysis will turn out to be very fruitful.

2)Apply the well-known technique of Kalman filtering. Extensive literature addresses this problem and therefore it will not be discussed herein.

Perhaps one might be keen on asking why I insist so much on applying gravity gradiometer techniques for terrestrial navigation systems since the up-to-date used systems can operate with better accuracy(one nautical mile per hour flight, approximately). The detail and accurate notion of the earth's gravity field is not of great importance for such applications. But the answer being straightforward, comeswith the question:what these gradiometer-unaided systems can do for space missions in cases of which the on-board platform travels through different, successive and completely unknown gravity fields? I believe, they can do nothing(strictly speaking).

A gradiometer-aided inertial navigation system turns out to be very profitable in many applications and it can surely upgrade the abilities of Mankind to explore the mystery which is here to stay, the distant cosmos.

Bibliography

Andreev, V.D. (1969) : Theory of Inertial Navigation, Israel Program for Scientific translations, Jerusalem.

Anthony, D. (1970) : Navigation Requirements for Dynamic Gravimetry by Means of Gradient Measurements, Edt. Kattner, W., Proc. of the Symp. on Dynam. Gravim., Fort Worth, Texas, Instr. Soc. of Amer., Pittsburgh.

Ausman, J. (1962) : Theory of Inertial Sensing Devices, Edt. Pittman, G. ("Inertial Guidance"), J. Wiley.

Ayres, F. (1962) : Matrices, Schaum's Series, McGraw-Hill.

Blais, J. (1977) : Inertia, Inertial Reference Systems and Physical Geodesy, Amer. Geoph. Union Annual Fall Meeting, San Francisco, California.

Brandley, C. (1970) : Identification of Vertical Deflections Utilizing an Inertial Navigator and Optimum Data Processing, Edt. Kattner, W., Proc. of the Symp. on Dyn. Gravim., Fort Worth, Texas, Instr. Soc. of Amer., Pittsburgh.

Britting, K. (1971) : Inertial Navigation Systems Analysis, J. Wiley.

Chandler, D. (1962) : Stable Platforms, Edt. Pittman, G. ("Inertial Guidance"), J. Wiley.

Chovitz, B. (1973), Lucas, J., Morrison, F. : Gravity Gradients at Satellite Altitudes, NOAA Tech. Rep., NOS 59, Rockwille, MD.

Denhard, W. (1977) : Aerial Profiling of Terrain System Design Phase Interim Report, R-1070, The Charles Stark Draper Laboratory, Cambridge, Mass.

Doukakis, E. (1976) : The Effect of Gravity Gradients in Point Coordination, M.Sc. Thesis, Oxford University, England.

Doukakis, E. (1978) : Remarks on Time and Reference Frames, Bull. Geod., Vol. 51, No. 4.

Fock, V. (1964) : The Theory of Space, Time and Gravitation, Pergamon.

Forward, R. (1972) : Geodesy with Orbiting Gravity Gradiometers, The Use of Artificial Satellites for Geodesy, Geophysical Monograph 15, The Amer. Geoph. Union, Washington, D.C.

Grafarend, E. (1972) : Three Dimensional Geodesy and Gravity Gradients, Rep. No. 174, Dept. of Geod. Science, The Ohio State University, Columbus, Ohio.

Grafarend, E. (1973) : Gravity Gradients and Three Dimensional Net Adjustments Without Ellipsoidal Reference, Rep. No. 202, Dept. of Geod. Science, The Ohio State University, Columbus, Ohio.

Grafarend, E. (1976, Schaffrin, B. : Equivalence of Estimable Quantities and Invariants in Geodetic Networks, ZFV, Heft 11.

Gorenstein, I.A. (1971) : Inertial Navigation Systems, AD-734630, NTIS, Ohio.

Gumert, W. (1970) Cobb, G. : Helicopter Gravity Measuring System, Edt. Kattner, W., Proc. of the Symp. on Dynam. Gravim., Fort Worth, Texas, Instr. Soc. of Amer., Pittsburgh.

Harter, G. (1962) : Error Analysis and Performance Optimization of Rocket Vehicle Guidance Systems, Edt. Pittman, G. ("Inertial Guidance"), J. Wiley.

Hammer, S. (1970) : The Anomalous Vertical Gradient of Gravity, Geophysics, Vol. 35, No. 1.

Heiskanen, W.A. (1967) Moritz, H. : Physical Geodesy, Freeman, San Francisco.

Heller, W. (1976) : Gradiometer-Aided Rapid Gravity System, Rep. SP-957, The Analytic Sciences Coorporation, Reading, MA.

Heller, W. (1976) Jordan, S. : Error Analysis of Two New Gradiometer-Aided Inertial Navigation Systems, J. of Spacecr. and Rock., Vol. 13, No. 6.

Hildebrant, R. (1974) Britting, K. Madden, S. : The Effect of Gravitational Uncertainties on the Error of Inertial Navigation Systems, J. of Inst. of Navig., Vol. 21, No. 4.

Hirvonen, R.A. (1971) : Adjustment by Least Squares in Geodesy and Photogrammetry, Frederic Ungar Publ., New York.

Hood, P. (1969) Ward, S.H. : Airborne Gravity Methods, Advances in Geophysics, Academic Press, New York.

Hotine, M. (1969) : Mathematical Geodesy, ESSA Monograph, Washington, D.C.

Joos, K. (1977) : Fehlermodel eines Inertialnavigationssystems für die Integrierte Navigation durch Ausgleichung nach der Methode der Kleinsten Quadrate, Z. Flugwiss. Weltraumforsch. 1, Heft 3.

Kachikas, G. (1962) : Guidance Kinematics, Edt. Pittman, G. ("Inertial Guidance"), J. Wiley.

Kachikas, G. (1969) : Error Analysis for Cruise Systems, Edt. Pittman, G. ("Inertial Guidance"), J. Wiley.

Kaula, W. (1970) : Application of Space and Astronomic Techniques to Solid Earth and Ocean Physics, Edt. Kattner, W., Proc. of the Symp. on Dynam. Gravim., Fort Worth, Texas, Instr. Soc. of Amer., Pittsburgh.

Kayton, M. (1962) : Fundamental Limitations on Inertial Navigation, Progress in Astronautics and Rocketry, Vol. 8, Academic Press, New York.

Klingenmeier, P. (1970) Meldrum, M. Metzger, E. : Error Sources in a Dynamic Gravimeter, Edt. Kattner, W., Proc. of the Symp. on Dynam. Gravim., Instr. Soc. of Amer., Pittsburgh.

Lipschutz, S. (1968) : Linear Algebra, Schaum's Series, McGraw-Hill.

McMillan, W.D. (1936) : Dynamics of Rigid Bodies, Dover, New York.

Meditch, J.S. (1969) : Stochastic Optimal Linear Estimation and Control, McGraw-Hill.

Meissl, P. (1970) : Probabilistic Error Analysis of Airborne Gravimetry, Rep. No. 138, Dept. of Geod. Science, Ohio State University, Columbus, Ohio.

Metzger, E. (1974) Jircitano, A. : Analysis of Real Time Mapping of Horizontal and Vertical Gravity Anomalies Aboard a Moving Vehicle such as an Aircraft, International Symposium on Applications of Marine Geodesy, Columbus, Ohio.

Moritz, H. (1968) : Kinematical Geodesy, Rep. No. 92, Dept. of Geod. Science, Ohio State University, Columbus, Ohio.

Moritz, H. (1971) : Kinematical Geodesy II, Rep. No. 165, Dept. of Geod. Cience, Ohio State University, Columbus, Ohio.

Moritz, H. (1974) : Some First Accuracy Estimates for Applications of Aerial Gradiometry, Rep. No. 209, Dept. of Geod. Science, Ohio State University, Columbus, Ohio.

Moritz, H. (1975) : Combination of Aerial Gravimetry and Gradiometry, Rep. No. 223, Dept. of Geod. Science, Ohio State University, Columbus, Ohio.

Ralston, A. (1960), Wilf, H. : Mathematical Methods for Digital Computers, J. Wiley.

Ramsayer, K. (1977) : Integrated Navigation Systems for Precise Positioning in the Deep Ocean, Marine Geodesy, Vol. 1, No. 1.

Reed, G. (1973) : Application of Kinematical Geodesy for Determining the Short Wave Length Components of the Gravity Field by Satellite Gradiometry, Rep. No. 201, Dept. of Geod. Science, Ohio State University, Columbus, Ohio

Rose, R. (1972), Nash, R. : Direct Recovery of Deflections of the Vertical Using an Inertial Navigator, IEEE Transactions on Geosciences and Electronics, Vol. GE-10, No. 2.

Savet, P. (1966) : Gravity Gradients: Their Intrinsic Representation and Applications, J. Spacecraft, Vol. 3, No. 7.

Savet, P. (1970) : New Developments in Gravity Gradiometry, Edt. Kattner, W., Proc. of the Symp. on Dynam. Gravim., Fort Worth, Texas, Instr. Soc. of Amer., Pittsburgh.

Scheid, F. (1968) : Numerical Analysis, Schaum's Series, McGraw-Hill.

Schultz, O. (1969), Winokur, J. : Shipboard or Aircraft Gravity Vector Determination by Means of a Three-Channel Inertial Navigator, J. Geophys. Res., Vol. 74, No. 20.

Schwarz, K.P. (1977) : Capabilities of Airborne Gradiometry for Gravity Estimation, Bolletino di Geodesia e Scienze Affini, Anno XXXVI, No. 2.

Sliv, E.I. (1972) : Applied Theory of Inertial Navigation, JPRS-57694, NTIS, Virginia.

Spiegel, M. (1968) : Mathematical Handbook, Schaum's Series, McGraw-Hill.

Spiegel, M. (1971) : Advanced Mathematics for Engineers and Scientists, Schaum's Series, McGraw-Hill.

Spiegel, M. (1972) : Statistics, Schaum's Series, McGraw-Hill.

Sprinsky, W. (1973) : Inertial Navigation Systems, Geodetic Science 763, Columbus, Ohio.

Thompson, L. (1965), Bock, R., Savet, P. : Gravity Gradients Sensors and their Applications for Manned Orbital Spacecraft, Science and Technological Series, Astronautical Society.

Time Magazine (1978) : Page 33, March 13.

Trageser, M. (1970a) : A Gradiometer System for Gravity Anomaly Surveying, The Charles Stark Draper Laboratory, R-588, Cambridge, MA.

Trageser, M. (1970b) : Gravity Gradiometer Program, The Charles Stark Draper Laboratory, C-3615, Cambridge, MA.

Trageser, M. (1971) : Gravity Gradiometer Briefing, The Charles Stark Draper Laboratory, C-3372, Cambridge, MA.

Trageser, M. (1972) : Gravity Gradiometer Status Report, The Charles Stark Draper Laboratory, C-3807, Cambridge, MA.

Trageser, M. (1975) : Feasibility Model Gravity Gradiometer Test Results, The Charles Stark Draper Laboratory, Cambridge, MA.

Williams, O. (1975) : Gradiometry: An Assessment of the State-of-the-Art, Bureau Gravim. Inter., Bull. d' Inform., No. 36.

Winter, H. (1974) : Stützung des Vertikalkanals einer Trägheitsplattform mit einem Barometrischen Höhenmesser, DGON-Symposium, Kreiseltechnik, Heidelberg.

Veis, G. (1963) : The Use of Artificial Satellites for Geodesy, North-Holland Publ., Amsterdam.

Appendix A: Computer Programmes Used

Simulation I und II

```
      PROGRAM DOUKAK
      DIMENSION A(19,3),B(19,19),C(3,3),ZW(19,1)
      REAL*8 A,B,C
      M=19
      N=3
      MM=19
      NN=19
      CALL ALESEN(A,M,N,0)
      CALL ALESEN(B,MM,NN,2)
      CALL MAMUL3(A,M,N,1,B,MM,NN,0,A,M,N,0,C,N,N,ZW,19,1)
      CALL ADRUCK(C,N,N,0)
      STOP
      END
      SUBROUTINE MAMUL3(A,MA,NA,KENNA,B,MB,NB,KENNB,C,MC,NC,KENNC,
     1R,MR,NR,ZW,MZW,SYM)
      DIMENSION A(MA,NA),B(MB,NB),C(MC,NC),R(MR,NR),ZW(MZW,1)
      REAL*8 A,B,C,FAK1,FAK2,FAK3,R,S,ZW
      INTEGER AZ,AS,CZ,CS,SYM
      LOGICAL KENNAL,KENNBL,KENNCL,SYML
      KENNAL=KENNA.NE.1
      KENNBL=KENNB.NE.1
      KENNCL=KENNC.NE.1
      SYML=SYM.EQ.1
      AZ=MA
      AS=NA
      IF(KENNAL) GOTO 41
      AZ=NA
      AS=MA
   41 CZ=MC
      CS=NC
      IF(KENNCL) GOTO 42
      CZ=NC
      CS=MC
   42 DO1 K=1,AZ
      DO2 I=1,CZ
      S=0.
      DO2 J=1,AS
      IF(.NOT.KENNAL) FAK1=A(J,K)
      IF(KENNAL) FAK1=A(K,J)
      IF(.NOT.KENNBL) FAK2=B(I,J)
      IF(KENNBL) FAK2=B(J,I)
      S=S+FAK1*FAK2
    2 ZW(I,1)=S
      J=1
      IF(SYML) J=K
      DO1 L=J,CS
      S=0.
      DO1 I=1,CZ
      IF(.NOT.KENNCL) FAK3=C(L,I)
      IF(KENNCL) FAK3=C(I,L)
      S=S+ZW(I,1)*FAK3
      R(K,L)=S
    1 IF(SYML) R(L,K)=S
      RETURN
      END
      SUBROUTINE AINVER(A,N)
C     SUBROUTINE MATRITZENINVERTIERUNG
      DIMENSION A(N,N)
      IF(N.NE.1) GOTO 11
      A(1,1)=1/A(1,1)
      GOTO 12
   11 DO10 K=2,N
      K1=K-1
      DO10 I=1,K1
```

```
   10 A(K,I)=0
      N1=N-1
      DO1 K=1,N1
      T=A(K,K)
      A(K,K)=1.
      K2=K+1
      DO2 I=K2,N
      S=A(K,I)/T
      DO3 J1=1,K
    3 A(I,J1)=A(I,J1)-S*A(K,J1)
      DO4 J2=I,N
    4 A(I,J2)=A(I,J2)-S*A(K,J2)
    2 CONTINUE
    1 A(K,K)=T
C     INVERSE MATRIX
      T=A(N,N)
      A(N,N)=1
      DO9 K=1,N
    9 A(N,K)=A(N,K)/T
      DO5 I1=1,N1
      I=N-I1
      T=A(I,I)
      A(I,I)=1.
      DO6 K=1,I
      S=A(I,K)
      I2=I+1
      DO7 J=I2,N
    7 S=S-A(I,J)*A(J,K)
    6 A(I,K)=S/T
C     UMSPEICHERN
      DO5 K=I2,N
    5 A(I,K)=A(K,I)
   12 RETURN
      END
      SUBROUTINE ADDSUB(KENN,A,MA,NA,KENNA,B,MB,NB,KENNB,C,MC,NC)
C     SUBROUTINE MATRITZENADDITION UND -SUBTRAKTION
      DIMENSION A(MA,NA),B(MB,NB),C(MC,NC)
      INTEGER AZ,AS
      LOGICAL KENNL,KENNAL,KENNBL
      KENNL=KENN.EQ.1
      KENNAL=KENNA.NE.1
      KENNBL=KENNB.NE.1
      AZ=MA
      AS=NA
      IF(KENNAL) GOTO 41
      AZ=NA
      AS=MA
   41 DO1 K=1,AZ
      DO1 I=1,AS
      IF(.NOT.KENNAL) FAK1=A(I,K)
      IF(KENNAL) FAK1=A(K,I)
      IF(.NOT.KENNBL) FAK2=B(I,K)
      IF(KENNBL) FAK2=B(K,I)
      C(K,I)=FAK1+FAK2
    1 IF(KENNL) C(K,I)=FAK1-FAK2
      RETURN
      END
      SUBROUTINE ADRUCK(A,M,N,KENN)
C     SUBROUTINE DRUCKEN EINER MATRIX
      DIMENSION A(M,N)
      REAL*8 A
      N1=N
      M1=M
      IF(KENN.NE.1) GOTO 43
      M1=N
      N1=M
   43 WRITE(6,101)M1,N1
  101 FORMAT(2I5/)
      DO 1 I=1,M1
      IF(KENN.NE.1) WRITE(6,102) (A(I,J),J=1,N1)
      IF(KENN.EQ.1) WRITE(6,102) (A(J,I),J=1,N1)
```

```
  102 FORMAT(1H ,5F14.4)
    1 WRITE(6,102)
      RETURN
      END
      SUBROUTINE  A L E S E N  (A,M,N,KENN)
C     LESEN EINER UNTEREN DREIECKSMATRIX
      DIMENSION A(M,N)
      REAL*8 A,X
      M1=M
      N1=N
      IF(KENN.NE.1) GOTO 51
      M1=N
      N1=M
51    IF(KENN.GE.2)GOTO 52
      DO 1 I=1,M1
      IF(KENN.EQ.1) READ(5,102)  (A(J,I),J=1,N1)
      IF(KENN.EQ.0)READ(5,102)(A(I,J),J=1,N1)
  102 FORMAT(8F10.4)
    1 CONTINUE
3     READ(5,102)X
      RETURN
52    DO 2 I=1,M
2     READ(5,102)(A(I,J),J=1,I)
       IF(KENN.EQ.3)GOTO3
       DO 4 I=1,M
       DO 4 J=1,I
4      A(J,I)=A(I,J)
      GOTO 3
      END
```

Programmes for the Quantization Error Studies

a.Variance-case

```
      PROGRAM TWINS1
      REAL*8 A,B,C
      READ(1,50) NN,DT,SA,TA,SG,TG
      SA2=SA**2
      SG2=SG**2
      CALL VAR(DT,SA2,SG2,TA,TG,NN)
50    FORMAT(I5,3F10.4,1F15.10,1F10.4)
      STOP
      END
      SUBROUTINE VAR(DT,SA2,SG2,TA,TG,NN)
      DO 5 IN=3,NN
      X=0
      DO 6 IM=2,INM
      DO 6 IR=2,INM
      X=X+DFLOAT((IN-IM)*(IN-IR))*DEXP(-DABS(DT*DFLOAT(IM-IR))/TA)
6     CONTINUE
      Y=0
      DO 7 IM=2,INM
      DO 7 IR=2,INM
      DO 7 IK=2,IM
      DO 7 IP=2,IR
      Y=Y+DFLOAT((IN-IM)*(IN-IR))*DEXP(-DABS(DT*DFLOAT(IK-IP))/TG)
7     CONTINUE
      S=DT**4*(SA2*X+SG2*Y)
      WRITE(6,100) IN,S
      WRITE(2,100) IN,S
100   FORMAT(10X,F15.4)
      RETURN
      END
```

b.Covariance case

```
      PROGRAM TWINS3
      REAL *8 DT,SA,TA,SG,TG,SA2,SG2,S
      DIMENSION S(20,20)
      READ(1,50) NN,DT,SA,TA,SG,TG
      SA2=SA**2
      SG2=SG**2
      CALL COVA(DT,SA2,SG2,TA,TG,NN,S)
      NNM2=NN-2
50    FORMAT(I5,3F10.4,1F15.10,1F10.4)
      WRITE(13,200)
      WRITE(6,200)
      WRITE(2,200)
200   FORMAT(1H1,12H VAR - COV =)
      DO 6 I=1,NNM2
      WRITE(13,100) (S(I,J),J=1,NNM2)
      WRITE(2,100) (S(I,J),J=1,NNM2)
      WRITE(6,100) (S(I,J),J=1,NNM2)
100   FORMAT(/5X,12(D10.2))
6     CONTINUE
      STOP
      END
      SUBROUTINE COVA(DT,SA2,SG2,TA,TG,NN,S)
      DIMENSION S(20,20)
      REAL *8 DT,SA,TA,SG,TG,SA2,SG2,S,X,Y
      DO 5 IN=3,NN
      DO 5 IS=3,NN
      X=0.
      INM=IN-1
      ISM=IS-1
      DO 6 IM=2,INM
      DO 6 IR=2,ISM
      X=X+DFLOAT((IN-IM)*(IS-IR))*DEXP(-DABS(DT*DFLOAT(IM-IR))/TA)
6     CONTINUE
      Y=0.
      DO 7 IM=2,INM
      DO 7 IR=2,ISM
      DO 7 IK=2,IM
      DO 7 IP=2,IR
      Y=Y+DFLOAT((IN-IM)*(IS-IR))*DEXP(-DABS(DT*DFLOAT(IK-IP))/TG)
7     CONTINUE
      S(IN-2,IS-2)=DT**4*(SA2*X+SG2*Y)
5     CONTINUE
      RETURN
      END
```